A Dynamic Modeling of Network-level Accumulation Heterogeneity

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Short summary

This study proposes a dynamic modeling framework that characterizes spatial heterogeneity as a key state variable in urban traffic systems. By using the dispersion parameter of the negative binomial distribution as a bridge, the framework captures the temporal evolution of spatial dispersion of link accumulation, interpreted as a proxy measure of network homogeneity, and its interaction with network accumulation. This relationship is further extended to describe the evolution of accumulation heterogeneity across links. The model is evaluated using simulation data under both uncongested and congested traffic conditions. Results suggest that the proposed framework can capture the dynamic evolution of network heterogeneity in both regimes. Moreover, the model provides a physically interpretable mechanism for explaining hysteresis phenomena and establishes a foundation for improved aggregate traffic modeling and control.
Keywords: Heterogeneity, Macroscopic Fundamental Diagram (MFD), Negative binomial distribution, Traffic hysteresis.

1 Introduction

The Macroscopic Fundamental Diagram (MFD) has emerged as an important tool for describing and controlling traffic dynamics at the network level. The seminal work of Geroliminis & Daganzo (2008) demonstrated the existence of a well-defined relationship between network accumulation and production in urban networks, enabling aggregate-level modeling of urban traffic systems. Subsequent studies further investigated the properties of well-defined MFDs and highlighted the importance of spatial homogeneity for maintaining stable network performance (Geroliminis & Sun, 2011a). Building upon this framework, MFD-based control strategies were developed to regulate network inflow and improve overall performance. For example, Keyvan-Ekbatani et al. (2012) proposed feedback-based gating strategies based on MFD dynamics, demonstrating the potential of the MFD as both a modeling and control tool for urban transportation systems.

However, subsequent studies revealed that the MFD is not always well-defined, particularly when spatial heterogeneity emerges within the network. Buisson & Ladier (2009) showed that increased heterogeneity in traffic conditions can significantly degrade the MFD. Similarly, Mazloumian et al. (2010) demonstrated that spatial variability of vehicle densities plays a critical role in determining urban network capacity. In addition, hysteresis phenomena have been widely observed in both urban and freeway networks. Gayah & Daganzo (2011) showed that clockwise hysteresis loops can emerge due to network instability in urban networks, while studies such as Geroliminis & Sun (2011b) demonstrated that hysteresis is also present in freeway networks. These findings indicate that aggregate traffic dynamics depend not only on total accumulation but also on the spatial distribution of congestion and the evolution of network states.

Recognizing the importance of heterogeneity and hysteresis, many studies attempted to improve traffic control strategies based on the MFD framework. Perimeter control strategies were developed to regulate traffic inflows and mitigate congestion. For example, Geroliminis et al. (2012) proposed an optimal perimeter control framework based on model predictive control. This line of research was further extended by studies such as Haddad & Shraiber (2014), Aalipour et al. (2018), and Gao et al. (2022), which aimed to improve the robustness and effectiveness of MFD-based control under complex traffic conditions.

In addition, multi-region modeling and partitioning approaches were introduced to better capture spatial heterogeneity. Ji & Geroliminis (2012) proposed spatial partitioning methods to divide large urban networks into more homogeneous regions. Building upon these ideas, Ramezani et al. (2015) proposed a dynamic modeling framework for heterogeneity evolution together with hierarchical control strategies. Related work, such as Keyvan-Ekbatani et al. (2015), further extended multi-region control strategies for large-scale urban systems.

Despite these advances, most existing studies focus on improving traffic control under heterogeneous conditions, rather than explicitly modeling the dynamics of spatial heterogeneity itself. In particular, while previous studies identified heterogeneity as a key factor affecting MFD performance (Mazloumian et al., 2010; Geroliminis & Sun, 2011b) and proposed dynamic control frameworks (Ramezani et al., 2015), the evolution of heterogeneity and its relationship with network-level traffic dynamics remain insufficiently understood.

Motivated by this gap, this study proposes a dynamic modeling framework that explicitly characterizes the evolution of spatial heterogeneity and investigates its role in explaining hysteresis phenomena and network-level traffic dynamics. By modeling heterogeneity directly, this work aims to provide new insights into aggregate traffic behavior and offer new directions for MFD-based traffic control strategies.

2 Methodology

To alleviate the limitations of traditional MFD-based models, we develop a framework that explicitly incorporates spatial heterogeneity into aggregate traffic modeling. While previous studies have highlighted the role of heterogeneity in MFD dynamics (Ramezani et al., 2015), it is typically treated as an observed outcome. Here, we introduce a dynamic representation of heterogeneity and integrate it into the aggregate modeling framework.

Instead of representing traffic states solely by network accumulation, we introduce an additional variable that captures the dispersion of link-level conditions. In this study, we first model the heterogeneity using a probabilistic approach for each time, which allows us to establish a direct connection between aggregate variables and the distribution of link-level traffic states. Then, based on this representation, we further develop a dynamic model to describe the temporal evolution of heterogeneity.

Negative Binomial Distribution

To represent the spatial heterogeneity of traffic conditions, we follow Ramezani et al. (2015) and model the distribution of accumulation across links at each time step using a negative binomial (NB) distribution. The general form of the NB distribution is given by

$$P(X = k) = \binom{k + r - 1}{k} p^r (1 - p)^k \tag{1}$$

where r is the number of successes and p is the success probability. Under this formulation, the mean and variance of the distribution are given by

$$\mu = r \cdot \frac{1 - p}{p}, \qquad \sigma^2 = r \cdot \frac{1 - p}{p^2} \tag{2}$$

This choice is motivated by the observed over-dispersion in urban networks, where the variance of link accumulation typically exceeds the mean.This phenomenon is further validated in our simulation results under both congested and uncongested scenarios (1c).

Let N_t denote the total network accumulation at time t, and let L be the number of links in the network. The mean accumulation across links is given by:

$$\mu_t = \frac{N_t}{L} \tag{3}$$

Under the NB assumption, the variance of accumulation over links at time t can be expressed as:

$$\sigma_t^2 = \mu_t + \frac{\mu_t^2}{r_t} \tag{4}$$

where r_t is the dispersion parameter of the NB distribution. The standard deviation σ_t is used as a quantitative measure of spatial heterogeneity.

The formula of standard deviation could be transferred as:

$$r_t = \frac{\mu_t^2}{\sigma_t^2 - \mu_t} \tag{5}$$

given μ_t and σ_t. This relationship allows heterogeneity to be interpreted in terms of the parameter r_t. In particular, smaller values of r_t correspond to higher levels of dispersion and stronger heterogeneity, while larger values indicate more homogeneous traffic conditions.

Notably, the NB assumption implicitly assumes that link accumulations are independent and identically distributed across the network. However, in real traffic systems, link conditions are spatially correlated due to congestion propagation and spillback effects, meaning that this assumption is not strictly satisfied. This represents a theoretical limitation of the current framework. Nevertheless, the Negative Binomial formulation still provides an effective aggregate representation of network heterogeneity. Incorporating spatial dependence remains an important direction for future research.

Dynamic Model of Heterogeneity

To capture the temporal evolution of heterogeneity, we model the dispersion parameter r_t as a dynamic process. Specifically, we assume that the logarithm of r_t follows a linear autoregressive structure:

$$\log \hat{r}_t = \beta_0 + \beta_1 \log \hat{r}_{t-1} + \beta_2 N_t + \beta_3 \Delta N_t \tag{6}$$

where $\Delta N_t = N_t - N_{t-1}$ represents the change in accumulation. $\hat{r}_t$ and $\hat{r}_{t-1}$ are predicted values. This formulation captures three key mechanisms: persistence in heterogeneity, dependence on the current congestion level, and responsiveness to changes in traffic conditions.

Having equations 5 and 6 means that we are able to capture the heterogeneity of link accumulation spatially and temporally. By iterating the equation 5, the model is applied to obtain the predicted trajectory of $\hat{r}_t$.

Then, reconstructing the equation 4 and plugging in the equation 3 to it resulting to:

$$\dot{\hat{\sigma}}_t = \hat{\sigma}_t - \hat{\sigma}_{t-1} = \sqrt{\frac{N_t}{L} + \frac{\left(\frac{N_t}{L}\right)^2}{\hat{r}_t}} - \sqrt{\frac{N_{t-1}}{L} + \frac{\left(\frac{N_{t-1}}{L}\right)^2}{\hat{r}_{t-1}}} \tag{7}$$

and the trajectory of the corresponding heterogeneity of link accumulation can be predicted.

Rather than treating the Negative Binomial distribution as a purely statistical representation, we provide a physical interpretation of its parameters. Spatial heterogeneity in urban traffic arises from uneven congestion formation across links, leading to smaller values of the dispersion parameter r_t. Here, if we arrange the equation 4, we can get

$$\frac{1}{r} = \frac{\sigma}{\mu} - \frac{1}{\mu} \tag{8}$$

The $\frac{\sigma}{\mu}$ is known as the coefficient of variation, which reveals the relative heterogeneity, and thus $\frac{1}{r}$ shows how much heterogeneity exceeds the homogeneity. Accordingly, r_t is interpreted as a proxy measure of homogeneity, while the associated probability parameter p_t, which increases monotonically with r_t, reflects the homogeneity tendency of traffic conditions. Modeling the dynamics of these parameters enables the framework to capture the evolution of spatial heterogeneity underlying hysteresis in network traffic.

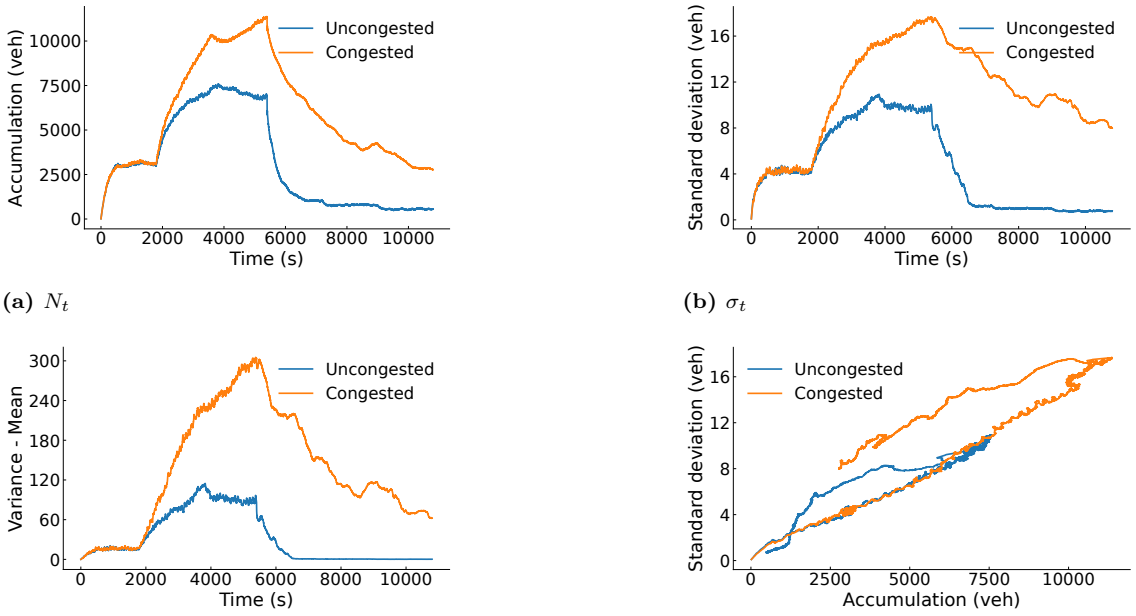
3 Results and Discussion

The proposed framework is evaluated using simulation data generated from the Barcelona network in Aimsun for 3 hours. Traffic states are obtained through Aimsun API outputs and processed using

Python to compute aggregate variables and link-level distributions. Two representative scenarios are considered: uncongested and congested traffic conditions, allowing comparison of heterogeneity dynamics under different network states.

Figure 1a and 1b illustrate the evolution of network accumulation and heterogeneity indicators under the two traffic regimes. Under uncongested conditions, network accumulation exhibits relatively stable dynamics with a sudden drop. These rapid decreases indicate that congestion does not persist and unloading occurs quickly, suggesting limited congestion propagation and weak spillback effects. Correspondingly, the standard deviation of link accumulation remains relatively low, reflecting more homogeneous traffic distribution across links. In contrast, the congested scenario exhibits stronger fluctuations in accumulation, accompanied by significantly larger standard deviation values, indicating increased spatial heterogeneity across the network.

Furthermore, Figure 1c shows the evolution of $(\sigma_t^2 - \mu_t)$. Smaller values are observed under uncongested conditions, while substantially larger values appear in the congested scenario, indicating increased spatial heterogeneity across links. The hysteresis patterns further illustrate these differences. As shown in Figure 1d, identical accumulation levels correspond to different levels of heterogeneity during loading and unloading phases. This observation confirms that network performance is influenced not only by total accumulation but also by the spatial distribution of vehicles.



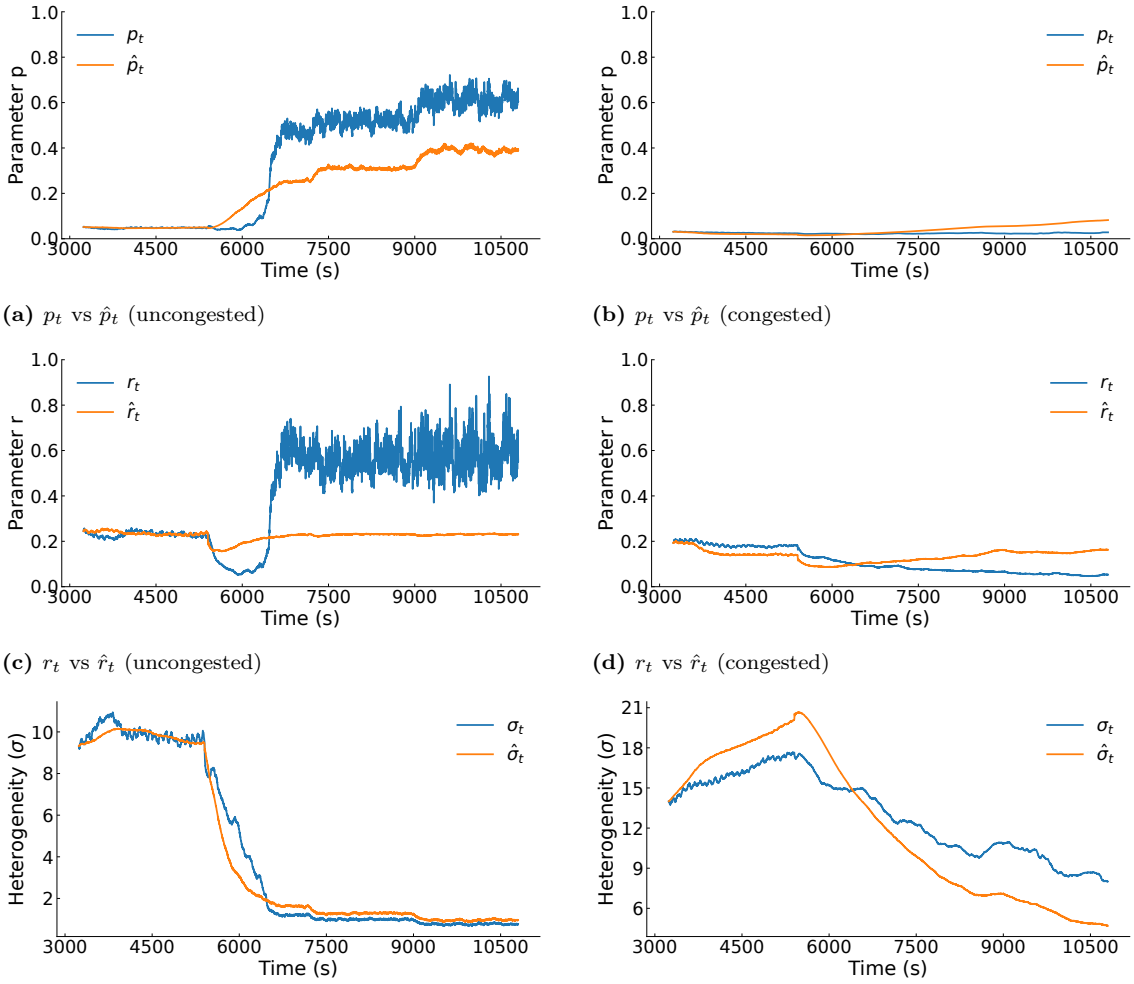
(a) N_t **(b)** σ_t **(c)** $\sigma_t^2 - \mu_t$ **(d)** Hysteresis

Figure 1: Comparison between uncongested and congested scenarios: (a) accumulation N_t, (b) standard deviation σ_t, (c) variance minus mean $(\sigma_t^2 - \mu_t)$, and (d) hysteresis patterns. The congested scenario exhibits larger dispersion and stronger hysteresis effects, indicating higher spatial heterogeneity.

Figure 2 evaluates the predictive performance of the proposed dynamic heterogeneity model. To assess the out-of-sample prediction capability, the three-hour simulation data are divided into two parts. The first 30% of the time series is used for model calibration, while the remaining 70% is reserved for pure prediction. As shown in Figure 2, the predicted trajectories of proxy measure of homogeneity r_t, the homogeneity tendency parameter p_t, and the heterogeneity measure σ_t are relatively close to the observed values under both uncongested and congested conditions. This agreement indicates that the proposed model successfully captures the temporal evolution of spatial heterogeneity.

A particularly important observation from Figure 2 is the consistent co-movement between the proxy measure of homogeneity r_t and the homogeneity tendency parameter p_t. In both uncongested and congested scenarios, increases in r_t are accompanied by increases in p_t, while decreases in r_t correspond to decreases in p_t. This consistent same-direction relationship reflects the intrinsic connection between these two parameters in the negative binomial representation of link accumulation. From a physical interpretation perspective, larger values of r_t indicate reduced dispersion,

corresponding to more homogeneous traffic conditions. Since the probability parameter p_t increases monotonically with r_t, the observed co-movement between these two parameters suggests that p_t can also be interpreted as an indicator of network homogeneity tendency.



(a) p_t vs $\hat{p}_t$ (uncongested)

(b) p_t vs $\hat{p}_t$ (congested)

(c) r_t vs $\hat{r}_t$ (uncongested)

(d) r_t vs $\hat{r}_t$ (congested)

(e) σ_t vs $\hat{\sigma}_t$ (uncongested, NMAE = 0.10)

(f) σ_t vs $\hat{\sigma}_t$ (congested, NMAE = 0.19)

Figure 2: Comparison of predicted and observed heterogeneity indicators under uncongested and congested conditions. The first column corresponds to the uncongested case, and the second column to the congested case. The figure compares the evolution of the homogeneity tendency parameter p_t, proxy measure of homogeneity r_t, and heterogeneity measure σ_t with their predicted counterparts.

Another important observation is that the model performs consistently under both uncongested and congested conditions. Although the congested scenario exhibits stronger fluctuations and higher heterogeneity levels, the predicted trajectories still match the trend of observed dynamics. The normalized mean absolute error (NMAE) of heterogeneity prediction remains relatively low in both scenarios, with 10% in the uncongested case and 19% in the congested case. These results indicate that the proposed dynamic model maintains robust predictive performance across different traffic scenarios.

Table 1: Estimated coefficients under uncongested and congested conditions

Condition	β_0	β_1	β_2	β_3
Uncongested	-0.007372	0.994973	0.000000	0.000238
Congested	-0.002985	0.998150	-0.000000	0.000242

Table 1 reports the estimated coefficients for both traffic conditions. In both cases, β_1 is close

to one, indicating strong persistence in heterogeneity dynamics. This suggests that spatial heterogeneity evolves gradually over time rather than changing abruptly. The coefficients β_2 and β_3 are relatively small, which can be attributed to the scale difference between explanatory variables and the log-transformed parameter $\log r_t$. Since the accumulation level N_t is on the order of thousands while $\log r_t$ varies within a much smaller range, the estimated coefficients appear numerically small. This observation suggests that future work may benefit from rescaling explanatory variables to improve numerical stability while preserving the physical interpretation of the model.

Overall, the results demonstrate that the proposed framework successfully captures the dynamic evolution of spatial heterogeneity and provides a physically interpretable mechanism for hysteresis phenomena. In particular, the consistent same-direction relationship between r_t and p_t aligns with their roles in the negative binomial distribution, where r_t represents the success times parameter and p_t denotes the success probability. This relationship provides an interpretable connection between statistical distribution parameters and traffic heterogeneity dynamics. By explicitly modeling these dynamics, the proposed framework improves the representation of network traffic behavior and provides a foundation for more accurate aggregate modeling and control applications.

4 Conclusions

This research develops a dynamic framework that explicitly models network heterogeneity to enhance macroscopic traffic representations. By connecting link-level distributions to network states via the negative binomial dispersion parameter, the model successfully captures the evolutionary trends of spatial variability over links. However, two critical areas for future refinement are identified. First, the current heterogeneity modeling assumption relies on the independent and identically distributed assumption for link accumulations. Given that spatial spillback creates dependencies between adjacent links, future iterations will explore relaxing this assumption to better reflect realistic topology. Second, the estimated coefficients β_2 and β_3 in table 1 are currently very small due to the significant scale difference between total network accumulation (N_t) and the log-transformed dispersion parameter ($\log r_t$). Future work will implement rescaling of these explanatory variables (e.g., normalizing N_t by network capacity) to ensure numerical stability and improve the sensitivity of the dynamic model.

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