Wardropian Cycles make assignment both optimal and fair.

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SHORT SUMMARY

We show how System Optimal assignment can be User Equilibrated and both Wardop principles can be reached at the same time. System Optimal traffic assignment is unfair as travel times of drivers using different routes connecting the same OD pair typically differ. Considering a finite sequence of days, however, the average travel times can be equilibrated for all the drivers on a given OD pair. We call such a sequence of daily assignments of drivers to routes a Wardropian Cycle and demonstrate that it always exists. We introduce analytical methods to identify such Cycles, along with a greedy algorithm which every day assigns users to routes so as to minimize the inequity. Cyclical User Equilibrium (that we introduce) allows to remove Price of Anarchy (and save of up to 670 hours of travel time in our Barcelona case).
Keywords: CAVs, game-theory, system optimal, traffic assignment, user equilibrium

1 INTRODUCTION

In principle, there are two classes of solutions to the traffic assignment problems. First, rational humans arrive at User Equilibrium (UE) where none of the drivers can individually benefit from changing their chosen routes. It is widely accepted, that such equilibrium arises in natural conditions when drivers do not coordinate their decisions and aim to maximize personal goals. However, UE typically leads to suboptimal system and the global performance (typically measured with total travel times or costs) can be improved. This, on the other hand, is reached at the System Optimal (SO) assignment, which yields minimal total travel costs. Through several individual concessions, coordination on route-choice process among travelers opens room for improvements at the global level. The difference between the UE and SO system performance, measured with the so-called Price-of-Anarchy (PoA) Youn et al. Youn et al. (2008) tells how much users could benefit (globally) when coordinating towards global optimum.
The two Wardrop principles have so far been considered irreconcilable: either the equity of uncoordinated rational utility maximizers entails suboptimal User Equilibrium, or fully compliant drivers following centralized orders collectively reach a global goal of System Optimum, regardless of individual costs. The advent of CAVs, potentially centralized (and/or collaborative) and compliant, motivates us to revisit the fundamentals of traffic assignment. Even though pure SO approaches were already proposed in Wang et al. Wang et al. (2018), in many contexts users are unlikely to fully comply to reach global optimum (minimize times or emissions), thus a new paradigm is needed. The first prerequisite is that the gains from coordination are not only global, but also individual, which is not guaranteed with classic SO approaches. Second, the egalitarian property of Nash, implicitly guaranteeing everybody a kind of social justice, should be preserved. In particular, paraphrasing Wardrop, no one departing with me from the same origin is supposed to arrive at the destination earlier than me. We show that, with a slight change of perspective, the above is doable.
We build our proposal on the following assumptions. First, central coordination of routing decisions may become possible in future systems with CAVs. Second, we extend the classical Wardopian perspective towards a sequence (cycle) of daily assignments. Third, we look for a new class of Equilibria obtained over a sequence of daily assignments. Departing from those, we develop an individual assignment policy on an OD pair which, by leveraging CAVs capabilities, allows SO to improve travel times against UE for all drivers over a period. We build on the SO solution and propose a sequence of daily assignments which eventually becomes User Equilibrated (mean times are equal). Figure 1 and example in Sec. 1 illustrate the general idea and Section 2 provides a formal introduction of the proposed Wardopian Cycles.

Illustrative example - proof of concept

Three travelers use capacitated routes to get from a common origin to a common destination departing at the same time. Uncoordinated, each of them travels 5 minutes. Suppose that if coordinated, it can be reduced

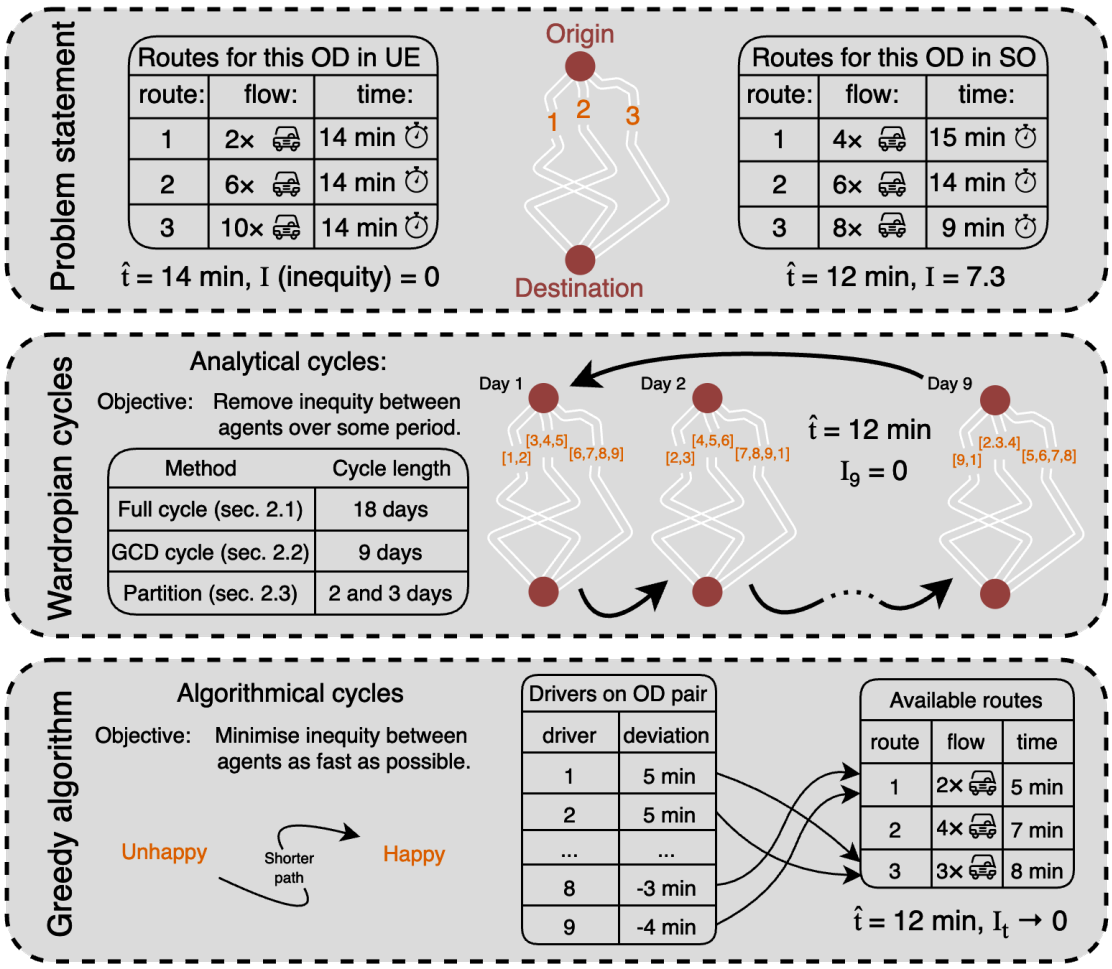


Figure 1: Methodology at glance: **1.** In comparison to User Equilibrium, System Optimal assignment is on average faster, but unfair. We measure this unfairness by Inequity (eq. 2). **2.** To achieve fairness in SO, for each OD pair, we create a periodic assignment rule to achieve equal average travel times after couple days, which we call *Wardropian cycles*. Apart from full cycles, that are oftentimes long, we propose two solutions to shorten those cycles. **3.** We design a greedy algorithm that, by assigning better off agents to longer routes day by day, decreases inequity of the system quicker, but not always to zero.

to 4 minutes: one travels 6 minutes and the remaining two travel 3 minutes. Why would the one traveling 6 minutes agree to that? We need either a dictator who will reduce social score for non-compliance, external incentives (e.g. mobility credit), or a different class of solutions. Here, the solution is a 3-day cycle: everyone for two consecutive days enjoys a faster route, followed with a one-day of travel on a longer route. This way, the assignment is both optimal (average travel time is 4 minutes) and fair (it is equal among travelers). We believe that such offer, if well communicated and realized, should be widely accepted.

Background

The cost of suboptimality of UE routing (Price-of-Anarchy Youn et al. (2008)) is paid not only by each individual drivers (arriving on average later) but also in externalities, like emissions, noise and congestion. In formal analyses, even in simple examples, the UE costs are up to 4/3 of SO ones Roughgarden & Tardos (2002), translating into millions of liters of fuel, tonnes of CO_2 and NO_2 globally. Reducing this *price* remained so far theoretical, as independent drivers were free to select their routing in the city and a centrally controlled alternative remain theoretical or dystopian only, this may change with the advent of CAVs.
After Wardrop Wardrop (1952) introduced the principles of System Optimum (SO) and User Equilibrium (UE), multiple attempts have been made to reconcile these approaches for fair and efficient traffic assignment. Most proposed methods focus on Constrained System Optimum, which balances efficiency and inequity better than UE but falls short of full SO efficiency. Like introducing individual assignment based on SO

which allows for an arbitrary deviation from UE Jahn et al. (2000), or applying a formulation based on Nash Welfare Lujak et al. (2015). Other approaches tried to achieve such balance by relaxing the User Equilibrium conditions, for example by reducing the number of feasible paths to more balanced set as in Zhou & Li (2012), or using a Bounded Rational User Equilibrium, proposed in Mahmassani & Chang (1987). We were inspired with the game-theoretical approaches, namely the mixed-strategies Haller (2023), which some similarities with cycles that we introduce.
While previous studies have extensively explored achieving System Optimum in traffic assignments, all attempts focus on balancing SO with UE, rather than guiding drivers toward SO by ensuring better average travel times. The idea of bridging SO and UE via the longer perspective is, up to our knowledge, novel. To our best knowledge, no prior work has considered inequity over a period while maintaining a complete System Optimum solution.

2 METHODOLOGY

Consider a flow of Q vehicles on an origin-destination (OD) pair. The global SO solution splits the total flow Q into flows Q_k along used routes r_k resulting in travel times t_k. This yields the average travel time $\hat{t} = \sum_k Q_k t_k / Q$ typically shorter than the UE[1].

Definition 1. *A system optimal* daily assignment *on a given day j is a matrix $A_j \in \mathscr{M}^{Q\times K}(\{0,1\})$ such that*

$$\sum_{k=1}^{K} [A_j]_{i,k} = 1 \text{ for every } i \in 1,\dots,Q. \tag{1a}$$

$$\sum_{i=1}^{Q} [A_j]_{i,k} = Q_k \text{ for every } k \in 1,\dots,K, \tag{1b}$$

A_j is a binary matrix, where $[A_j]_{i,k} = 1$ indicates that driver i is assigned to route k. Condition (1a) imposes that each driver is assigned to exactly one route and condition (1b) states that the numbers of drivers on each route correspond to SO.

Definition 2. *For a given day j and daily assignment A_j, we define the vector of travel time deviations D_j by*

$$\mathbf{D}_j \quad = \quad A_j \begin{bmatrix} t_1 \\ t_2 \\ \dots \\ t_K \end{bmatrix} - \begin{bmatrix} \hat{t} \\ \hat{t} \\ \dots \\ \hat{t} \end{bmatrix},$$

where t_k is the path k travel time and $\hat{t}$ is the average travel time.

Intuitively, D_j denotes the deviation, on day j, from the average travel time for each driver.

Definition 3. *For a given sequence of daily assignments $(A_1,\dots,A_J)$, inequity up to day J is given by*

$$I_J = \left(\sum_{j=1}^{J} D_j \times \sum_{j=1}^{J} D_j \right) / Q. \tag{2}$$

Which is basically just the sum of the squared daily deviations.

Definition 4. *Wardropian cycle is a finite sequence of daily assignments $C_J = (A_1,\dots,A_J)$ such that $I_J = 0$.*

Remark 1. *A Wardropian cycle emerges when all travelers experience the same total travel time over a period. We note that $I_J = 0 \iff \sum_{j=1}^{J} D_j = 0$ since all elements in the dot product in the definition of I_j are squared (non-negative).*

In the following subsections, we prove existence of such cycles and attempt to decrease their length. Shorter cycles offer a practical advantage: they make it easier to encourage drivers to follow them, as drivers experience improvements sooner, enhancing their willingness to comply. We start with an upper bound on the cycle length, which is solely based on SO traffic distribution. (Sec. 2). Next, we incorporate route flow information (Sec. 2). We then use both route flow and route times to propose an algorithm to minimize the required cycle length (Sec. 2). Finally, we attempt to provide a new paraphrase of Wardrop principles and introduce a Cyclical Wardrop Equilibrium (Sec. 2).

[1]While $\hat{t}_{UE} \geq \hat{t}$ for most of OD pairs in the system it is not guaranteed for all of them. For some $\hat{t}_{UE} < \hat{t}$ and our approach fails: it guarantees equity of travel times, yet no benefits. Users of such OD pairs (17.5% in our Barcelona case-study) need to be incentivised differently to comply.

Upper boundary - full cycles

First, we show that for any OD pair of flow Q, there exist a *Wardropian Cycle* of length Q.

Proposition 1. *Let P be the permutation matrix defined as*

$$P = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

For any initial assignment A_1, the sequence of daily assignments $A_1, A_2, \dots A_Q$ where

$$A_{j+1} = P^j A_1 \text{ for } j = 1, \dots, Q-1. \tag{3}$$

is a Wardropian cycle of length Q.

Remark 2. *The permutation matrix P shifts user assignment by one, i.e. given that travelers are assigned to roads $r_{i_1}, r_{i_2}, \dots, r_{i_Q}$, on the following day they are assigned to roads $r_{i_2}, r_{i_3}, \dots, r_{i_Q}, r_{i_1}$, respectively.*

Proof of Proposition 1. Let $T = [t_1, \dots, t_K]^T$. The vector of total travel times of drivers can be expressed as

$$\left(\sum_{j=1}^{Q} A_j\right) T = \begin{bmatrix} Q_1 & \dots & Q_K \\ & \ddots & \\ Q_1 & \dots & Q_K \end{bmatrix} T = \begin{bmatrix} \sum_{i=1}^K Q_i t_i \\ \dots \\ \sum_{i=1}^K Q_i t_i \end{bmatrix} = Q \begin{bmatrix} \hat{t} \\ \hat{t} \\ \dots \\ \hat{t} \end{bmatrix}.$$

After Q days, each driver will be assigned to route k exactly Q_k times, hence their mean travel time will be equal to $\hat{t}$. □

While this approach ultimately achieves equity, it has certain drawbacks. Before completing the full cycle, some drivers may experience significant inequity, as the proposed assignment could place them on slower routes for an extended period before switching to faster ones. This issue can be mitigated by adjusting the permutation order, allowing for a more balanced distribution of deviations throughout the cycle.

Reduction by greatest common divisor.

Now, we show that a cycle can be shorter than Q. Instead of permuting the cycle by one daily (τ_1), we can use greatest common divisor of path flows $M = \mathrm{GCD}(\mathrm{Q}_1, \mathrm{Q}_2, \dots, \mathrm{Q}_\mathrm{k})$ and shift drivers by this number of steps everyday.

Proposition 2. *For any initial assignment A_1, we define daily assignments $A_2, \dots A_{Q/M}$ by*

$$A_{j+1} = (P^M)^j A_1 \text{ for } j = 1, \dots, Q/M - 1. \tag{4}$$

The sequence $A_1, \dots A_{Q/M}$ is a Wardropian cycle of length Q/M.

Proof. Similarly to proof of Proposition 1, we calculate the total travel time:

$$\left(\sum_{j=1}^{Q/M} A_j\right) T = \begin{bmatrix} Q_1/M & \dots & Q_K/M \\ & \ddots & \\ Q_1/M & \dots & Q_K/M \end{bmatrix} T = \begin{bmatrix} \frac{1}{M}\sum_{i=1}^K Q_i t_i \\ \dots \\ \frac{1}{M}\sum_{i=1}^K Q_i t_i \end{bmatrix} = Q/M \begin{bmatrix} \hat{t} \\ \hat{t} \\ \dots \\ \hat{t} \end{bmatrix}$$

Instead of traveling each route Q_k times, each driver is assigned there exactly Q_k/M times. Since, from definition, for every k this value is an integer, we construct a Wardropian cycle. □

While $M = 1$ brings us back to the solution from Section 2, in real urban networks often $M > 1$ which lead to significantly reduced cycle lengths.

Reduction by flow partition

The next step in the reduction of a Wardropian cycle length incorporates not only route flows, but also specific travel times on each route. It relies on splitting flow Q of each OD pair, into groups $J_1, \ldots, J_S$ of equal mean travel time. Unfortunately, dividing a set of routes with time and flow into subsets of equal mean time is *NP*-hard, which can be proven as a direct reduction from a Subset Sum Problem.
Now we no longer try to arrange every driver on the same cycle, instead we create several, shorter ones that all have equal average time, but not necessarily equal length, nor route set. We note that we can group routes with equal travel times together, as it is irrelevant whether a driver takes one route or another — the key factor is the travel time itself. From this time on, we can assume that route times are pairwise different. We can assume, that instead of each route r_k of flow Q_k, we have a set of Q_k unit routes (routes of flow 1) of equal time. This leads to a multiset, where each route r_k is present Q_k times. Hence, we can focus solely on travel times while the flow information is incorporated in the multiset structure. Such representation allows us to divide a route flow into several sub-cycles, while preserving the SO assignment. We want to partition the multiset of size Q into subsets J_s, where route r_i (of flow 1, present Q_k times in the multiset) has time $t_i, i \leq Q$, so that

$$\sum_{j \in J_s} \frac{t_j}{|J_s|} = \sum_{i \in [Q]} \frac{t_i}{Q} = \hat{t} \quad \forall s \in S \tag{5}$$

If such partition is found, any Wardopian cycle of daily assignments can be applied within the group. We argue that a Wardropian cycle can be constructed for each group separately leading to a fair assignment for this OD pair. This holds because, every driver in a Wardropian cycle achieves mean travel time equal to mean daily travel time, so will have travel time equal to the mean time of their partition J_s, hence equal for every driver on this OD pair.

Greedy algorithm to minimize inequity

The analytical approaches, while satisfactory (they guarantee that cycles exists and are finite), identify long cycles (up to few weeks, or months of commute as we will demonstrate in the results). To operationalize it and not rely on properties which are not granted (like high GCD of flows), we propose a heuristic approach to approximate the solution for the Wardropian cycle problem. While our method does not guarantee equilibration of mean travel times, it minimizes the Inequity in the system. The main rule of this algorithm is as follows: drivers experiencing the longest travel times so far are assigned to the shortest routes tomorrow. This approach is both fast to compute and effective.
Each day we sort drivers in decreasing order by sum of their travel time deviation (eq. 2), pairing them to corresponding route in order (see fig.1c). The drivers most favored so far are assigned to longer routes and vice versa, which accelerates the convergence.
Intuitively, each day we order drivers by total travel time up to this day in decreasing order and pair them with route on corresponding position in the set of unit routes sorted by increasing travel time. That means that on the following day, drivers that were unflavored previously will be compensated and vice versa.
Starting with any assignment A_1, we define recursively $A_{j+1} = P_j A_j$. We find permutation matrix P_j as follows. First, we calculate $\hat{D}_j = \sum_{j=1}^{J} D_j$. Then, we find a permutation π_j of the vector $\hat{D}_j$ such that $\pi_j(\hat{D}_j)$ is decreasing (i.e. each subsequent element of the vector is weakly smaller). We denote $\pi_{j,l}$ the l-th element of this permutation. We define mapping $T_j : \{1, \ldots, Q\} \ni l \to k : \sum_{j \leq k} Q_{j-1} < l \leq \sum_{j \leq k} Q_j, k \in \{1, \ldots, K\}$, where $Q_0 = 0$. Ones in P_j are on positions $\{(\pi_{j,l}, T_j(l)) : l \in \{1, \ldots, Q\}\}$.
With such approach we may not reach the full equity (as we will demonstrate) thus, the cycles are not purely Wardropian anymore (compare Def.4). Which, however, allows to can control the cycles' lengths. We can identify the best cycles not longer than given days or cycles of maximal inequity (how many days we need to reduce inequity to a certain value).
Moving forward to system-wide analyses, we now normalize the inequity $\bar{I}$ as:

$$\bar{I}_J = \frac{\left(\sum_{j=1}^{J} D_j\right)^2}{QI_1}. \tag{6}$$

Which starts with 1, on day one, and our algorithms aims to minimize it. Using this notion, we denote *tolerance threshold* α and *minimal period length* $\mathbf{T}$ (with the tolerance threshold α) as:

$$\mathbf{T} := \min_j \{\bar{I}_j \leq \alpha\}. \tag{7}$$

If an algorithm achieves $I_j = 0$ for a certain j and an OD pair, we say that it *converged*. When a method that does not converge, yet it reduces the normalized inequity and finds a minimal period length at tolerance threshold α, we say that it *quasi-converges* (at the tolerance threshold α).

Cyclical User Equilibrium

We attempt to provide here a simple paraphrase of Wardop principles adequate for our introduced cycles, which we hope will turn out useful in following research.

Definition 5. *Drivers on a given OD pair are in Cyclical User Equilibrium, if no driver has incentive to decline the cyclic order of assignment and force the return to the UE assignment.*

This happens when for given OD pair SO assignment is better than UE and cyclic rule forms a Wardropian Cycle.

3 RESULTS AND DISCUSSION

To scrutinise properties of the proposed analytical and heuristic approaches, we construct an experimental framework. We look for Wardopian Cycles for 2078 OD pairs in Barcelona[2]. We use the established python framework by Bettini Bettini (2021) to compute static SO and UE assignments. Our generic and reusable python functions[3] compute analytical and algorithmic cycles. As argued, the finite Wardropian cycles exists in Barcelona. We analyse their length, convergence and resulting inequity. Expectedly, the first analytical approach, while guaranteeing a perfect convergence, produces long cycles. The GCD-based method improves results, yet for many OD pairs the inequity level stays high for a long period. Conversely, our heuristic quickly reaches satisfying levels of the travel time imbalance, yet sometimes does not achieve a complete convergence.

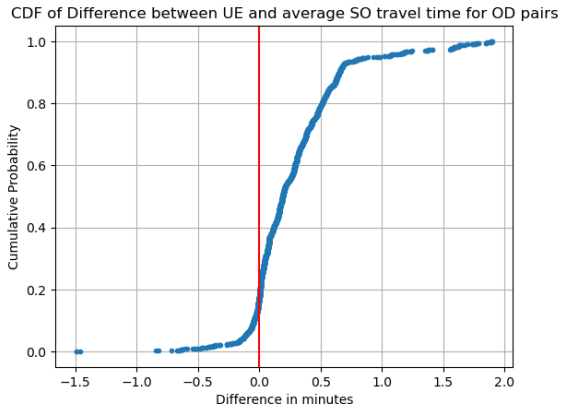


Figure 2: For most OD pairs, system optimal assignment is profitable, but in this example, 17.5% of OD pairs are worse off if SO assignment is applied.

First, we show the gains of moving from UE to SO. Applying the assignment that minimizes total travel time, improves the system globally by 3%, which translates to 670 hours of travel time. Fig. 2 shows how benefits are distributed across the OD pairs. One can notice that not all OD pairs benefit from SO, yet for 82.5% of OD pairs users will benefit from complying to our cyclical assignment and on average arrive faster.
Then, we report the distribution of cycle lengths computed with analytical methods for 2078 OD pairs in Barcelona in Fig. 3. As we see the cycles exist, yet are often too long to be acceptable for drivers.

Table 1: Cycle length statistics for Full cycles and GCD Cycles Methods. GCD optimisation improves results, but several cycles are still too long to accept.

Metric [days]	**Max**	**Mean**	**Median**	**Std Dev**	**75th Percentile**	**95th Percentile**
Full cycles (sect. 2.1)	777	31.37	16	52.06	34	100.3
GCD cycles (sect 2.2)	777	25.94	11	47.81	28	91

We conclude with the results from the heuristic algorithm. For most OD pairs, our method improves the

[2]The network topology and demand obtained from Stabler (2023)
[3]Available at a public repository - Hoffmann et al. (2024)

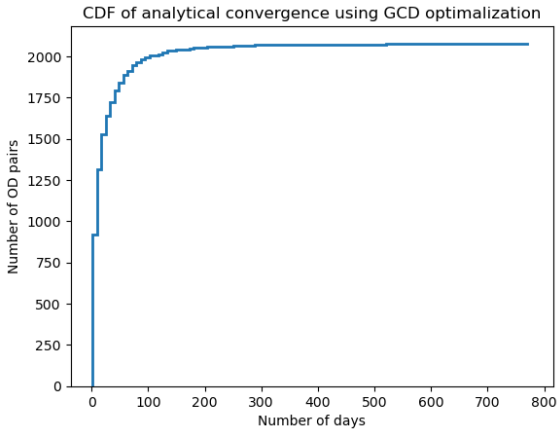


Figure 3: CDF graph of analytical convergence with GCD optimalization method. Although most cycles are short, there is a long tail with cycles as long as 800 days. Such cycles are impossible to implement in a real network.

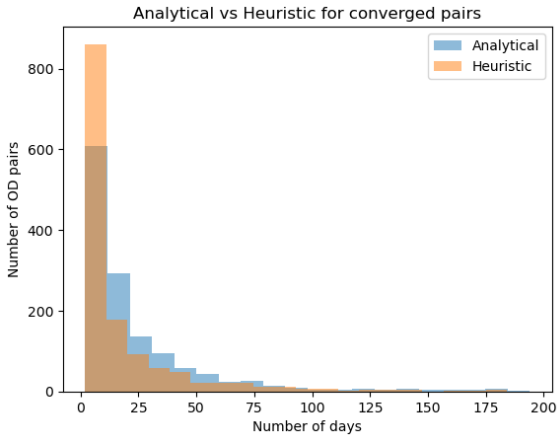


Figure 4: Cycle lengths in analytical and heuristic solutions. We included only pairs that converge for both solutions in a cycle shorter than 200 days for visibility.

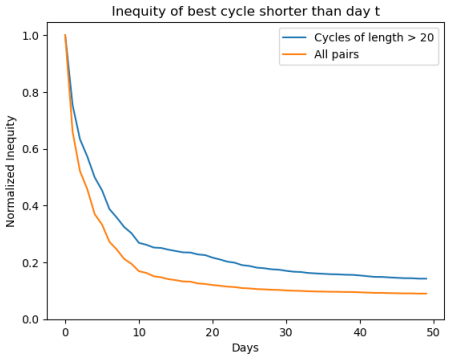
analytical solutions as presented in Fig. 4.
Although perfect inequity may not be fully achieved, our algorithm significantly reduces inequalities within just a few days. In five days, the total inequity (eq. 2) is more than halved and stabilizes soon afterwards (Figure 5 (a)) As a performance metric, we use minimal inequity over the first t days, as this represents the best cycle approximation we can create having at most t days. As can be seen in Table 2 and Figure 5 (b), we can offer reduction of inequity by as much as 88% in 20 days.

Table 2: Summary Statistics for Minimum Normalized Inequity Across Timeframes

Metric	**Max**	**Mean**	**Median**	**Std Dev**	**75th Percentile**
Ininitial Inequity	1.0	1.0	1.0	0	1.0
Min Inequity (5 days)	1.0	0.370	0.279	0.355	0.500
Min Inequity (10 days)	1.0	0.195	0.148	0.247	0.275
Min Inequity (20 days)	1.0	0.124	1.08×10^{-4}	0.200	0.203
Min Inequity (full simulation)	1.0	0.079	6.89×10^{-26}	0.153	0.134



(a) Inequity (eq. 2) in the system over time averaged over all OD pairs



(b) Minimal norm. inequity averaged over all OD pairs up to a given day

Figure 5: Inequity in the system decreases rapidly. While overall inequity stabilises after several days, on average, OD pairs reach a day with inequity as low as one tenth of initial value in just two or three weeks of simulation.

4 Conclusions

Our proposed method bridges the gap between User Equilibrium and System Optimum, which is of great importance for efficiency of our future transport systems. Only in Barcelona we may reduce the total travel time by 962 468 minutes in a single peak-hour. Unlike previous approaches, it achieves both optimality and equity. Numerical results demonstrate the feasibility of this approach, even in city-scale systems and within short time frames. The method is both practical and driver-beneficial, making it a promising candidate for real-world traffic management. Its applicability will further increase with the rise of CAVs, paving the way for more efficient and fair traffic systems in the coming years.We hope, that our idea can provide a new paradigm of traffic assignment and potentially remove price of anarchy in future systems.

Acknowledgements

This research is financed by the European Union within the Horizon Europe Framework Programme, ERC Starting Grant number 101075838: COeXISTENCE.

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