Causal Inference of Longitudinal Behavioral Responses to Mobility Interventions

Benjamin Cottreau*[1,2], Liam Solus[1], Leizhen Wang[2,3], and Zhenliang Ma[2]

[1]Department of Mathematics, KTH Royal Institute of Technology, Stockholm, Sweden
[2]Transport Planning Division, KTH Royal Institute of Technology, Stockholm, Sweden
[3]Department of Data Science and Artificial Intelligence, Monash University, Melbourne, Australia

Short summary

Public Transport Networks (PTN) are complex dynamical systems in which supply and demand continuously interact. Mobility interventions are introduced to address market failures and critical operating conditions, such as congestion, yet their behavioral effects are often difficult to evaluate causally. The Box-Tiao model provides a useful econometric framework for intervention analysis, but offers limited insight into causal mechanisms and heterogeneous responses over time. This study proposes a flexible multivariate causal learning framework based on graphical models to dynamically estimate intervention effects at both the population and individual levels. The framework generalizes the classical Box-Tiao model, which can be recovered as a marginal case. We apply the approach to the public transport fare incentive program introduced in Hong Kong in 2014. The results show that the proposed method can identify intervention effects, estimate the associated parameters, and uncover the direct influence of user characteristics on incentive adoption.

Keywords: causal inference, graphical model, public transport, intervention, incentive

1 Introduction

Mobility patterns are heavily time-dependent: working hours, school holidays, and other calendar features drive mobility demand. In this context, mobility systems often face capacity limits that conflict with socioeconomic efficiency. In this regard, mobility research has extensively focused on solving the induced problems, such as congestion or delays. This implies a range of policies that either focus on supply (eg. increase or diversify mobility supply, dynamically adapt supply) or demand (eg. provide information or incentives to influence individual mobility behaviors).

Among demand-oriented solutions, fare incentive programs seek to provide fare discounts to users, at strategic times and locations to redistribute the demand in space and times. Several works have investigated fare incentive programs through choice modeling (Zhou et al., 2022), data-driven analysis (Wang et al., 2025), and static causal graph models(Wu et al., 2025), but these works do not deal with the dynamic tuning and scalable implementation of such fare policies. Therefore, there is a need to formulate a modeling framework that answers the following questions: *To what extent policy interventions, such as fare incentives, achieve the desired outcome? How can we model the multivariate effects of policy interventions on longitudinal behaviors?*

The paper provides a data-driven, multivariate framework to dynamically assess intervention effects on Public Transport Networks (PTN). It relies on a causal discovery and causal inference, using graphical models (Lauritzen, 1996). Graphical models allow for the incorporation of observed user features, which are used to define individualized incentive programs. The basic principles of causal graphical modeling allow recovering conditional treatment effects, corresponding to intervention effects on individualized subpopulations. The unique approach of this work is the introduction of a parametric dynamic graphical model that allows us to (1) identify interventions effects, (2) estimate the associated parameters and (3) uncover the causal structure between user characteristics and incentive adoption.

2 Methodology

A common practice in economics is to model the evolution of a response variable over time by estimating its expectation $\mu_m^{(t)}$ at each time t with the help of sample mean estimates for the response variable $X_m^{(t)}$ at time t. The Box-Tiao approach provides estimators for the effects of a perturbation of the law of $X_m^{(t)}$ using the time series $\mu_m^{(t)} = \mathbb{E}[X_m^{(t)}]$ (Box & Tiao, 1975). This gives a simple and effective method for giving quantitative answers to questions such as, *How much did the incentivization program impact the response over time?* However, the Box-Tiao model is definitively *univariate*, and does not allow for the incorporation of features describing mobility behavior and also lack of causality-based effect assessment. This means that the method is inadequate to answer questions guiding *individualized* incentivization policies, such as, *Given that the traveler has a relatively flexible schedule, how much did the incentivization impact the response?*

We introduce a causal graphical model to give a multivariate extension of the Box-Tiao model with a explicitly embedded causal mechanism that allows us to causally estimate individualized incentivization effects over time, while recovering the classic Box-Tiao approach as a marginal time series.

a) Graphical Modeling

Modeling framework - Let $X_j^{(t)}$ be the random variables corresponding to a set of m features describing mobility behaviors over time, including the response variable $X_m^{(t)}$. We assume the data is distributed with respect to a Directed Acyclic Graph (DAG) $G = ([m], E)$. The j-th random variable is linked to its parents $\mathrm{pa}_G(j) \subset [m]$ and are related such that:

$$X_j^{(t)} = \sum_{k \in \mathrm{pa}_G(j)} \lambda_{kj} X_k^{(t)} + \varepsilon_j^{(t)}, \quad \varepsilon_j^{(t)} \sim \mathcal{N}(\rho_j^{(t)}, \omega_j^{(t)}) \tag{1}$$

where $\lambda_{kj} \in \mathbb{R}$ are the direct effects, $\rho_j^{(t)} \in \mathbb{R}$ is the error mean, $\omega_j^{(t)} \in \mathbb{R}_{>0}$ is the error variance and the $\varepsilon_j^{(t)}$ are mutually independent. A choice of parameters establishes a distributional law for the random vectors $X^{(t)} = (X_1^{(t)}, \ldots, X_m^{(t)})$, $t \geq 0$. The structure of the graph is static, meaning that there is no creation or removal of an edge over time. In this framework, the sample mean $\mu_m^{(t)}$ of the response variable $X_m^{(t)}$ at time t has an induced parameterization encoding how it is effected by the features. Let $\mathcal{P}(k, m)$ denote the set of directed paths $P = k \to i_1 \to i_2 \to \cdots \to m$ from k to m in G. Then $\mu_m^{(t)} = \mathbb{E}[X_m^{(t)}]$ is parametrized as:

$$\mu_m^{(t)} = \rho_m^{(t)} + \sum_{k=1}^{m-1} \sum_{P \in \mathcal{P}(k,m)} \rho_k^{(t)} \prod_{k \to i \in P} \lambda_{ki} \tag{2}$$

Since the parameters λ_{ki} in (1) encode the direct effect of X_k on X_i, summing over the products $\rho_k^{(t)} \prod_{k \to i \in P} \lambda_{ki}$ as in (2) encodes the effects of the features on the mean variable $\mu_m^{(t)}$. In this way, the parameterization of $\mu_m^{(t)}$ we give in (2) allows us to encode the effects of the mobility behavior features $X_1, \ldots, X_{m-1}$ on the mean of the response X_m.

One way to parameterize the effects of an intervention at time t_i in this model is to assume

$$\varepsilon_j^{(t)} = e^{-\beta(t-t_i)} \varepsilon_j^{(t_i)} + (1 - e^{-\beta(t-t_i)}) \varepsilon_j^*, \qquad t \geq t_i \tag{3}$$

for some $\beta \in \mathbb{R}_{>0}$ and $\varepsilon_j^* \sim \mathcal{N}(\rho_j^*, \omega_j^*)$ parameterizing a limiting distribution $X^* = (X_1^*, \ldots, X_m^*)$ as in (1). In this way, observations drawn from the joint distribution $X^{(t)} = (X_1^{(t)}, \ldots, X_m^{(t)})$ for t following the time of intervention t_i are a mixture of the behavior just before intervention and the eventual limiting behavior modeled by X^*. The parameter β governs the rate of progression towards this limit over time.

The goal of this work is to connect the classic (univariate) Box-Tiao models for the mean $\mu_m^{(t)}$ of the response $X_m^{(t)}$ to the graphical model parameterization (1) and therefore introduce causal relationships between explanatory and response variables. As we will see below, the Box-Tiao models for the time series $\mu_m^{(t)}$ can be recovered as a marginal time series for the response $X_m^{(t)}$

in the *multivariate* time series $X^{(t)} = (X_1^{(t)}, \ldots, X_m^{(t)})$ given by (1) with intervention modeled as in (3). This allows us to extend the classic Box-Tiao models to recover individualized incentivization effects by computing conditional expectations in $X^{(t)}$, which are easily computed using classic techniques from graphical modeling. For instance, in the following, Maximum Likelihood Estimates (MLE) of λ, ρ, ω and therefore μ are computed at each time step as described in Boege et al. (2025).

Causal discovery - Retrieving the causal structure of the data given by the DAG $G = ([m], E)$ is often referred to as *causal discovery*. It consists of finding the conditional independence relationship between variables given a data distribution. According to relationships, there exists a class of Partially Directed Acyclic Graphs (PDAG), that describes the causal and probabilistic dependence structure of obtainable from the observed data.

Several algorithms can be used to perform this task, and the reader might want to refer to Rios et al. (2025) for a thorough benchmark of existing methods. In this work, we use Greedy Equivalence Search (GES), which is one of the most used causal discovery algorithms (Maxwell Chickering, 2002). The principle is the following: GES starts with an empty graph and iteratively adds, and then removes, edges to maximize a score function, like Bayesian Information Criterion (BIC).

b) Dynamic Integration of Interventions using the Box-Tiao model

First-order difference equation - Box-Tiao models are originally studied in the context of time series intervention (Box & Tiao, 1975). These objects are useful for a range of fields such as electrical engineering, control engineering, and economics. In this perspective, economic intervention is seen as an external stimulus that influences the observed outcome, i.e. the response variable X_m. Let s be the step function that represents the presence of an intervention defined as:

$$s^{(t)} = \begin{cases} 0 & \text{if} \quad t < t_i \\ 1 & \text{if} \quad t \geq t_i \end{cases} \tag{4}$$

where t_i is the time of intervention. The time series of $\mu_m^{(t)}$, which represents the mean of the response variable $X_m^{(t)}$, can be written as a difference equation (Box & Tiao, 1975):

$$\mu_m^{(t)} = \delta_1 \mu_m^{(t-1)} + w_0 s^{(t)} + n_m^{(t)} \tag{5}$$

where $w_0 \in \mathbb{R}$, $\delta_1 \in (-1, 1)$, and $n_m^{(t)}$ is an Seasonal Auto-Regressive Integration Moving Average process (SARIMA) process of orders $p, d, q, P, D, Q \in \mathbb{N}$, with AR coefficients $\phi_1, ..., \phi_p$, seasonal AR coefficients $\Phi_1, ..., \Phi_P$, MA coefficients $\theta_1, ..., \theta_q$ and seasonal MA coefficients $\Theta_1, ..., \Theta_Q \in \mathbb{R}$.

Solutions - When solving 5 as shown in Appendix A1, the time series becomes:

$$\mu_m^{(t)} = \begin{cases} \mu_m^{(t_i)} + n_m^{(t)} & \text{if} \quad t < t_i \\ \mu_m^{(t_i)} e^{-\beta(t-t_i)} + \kappa(1 - e^{-\beta(t-t_i)}) + n_m^{(t)} & \text{if} \quad t \geq t_i \end{cases} \tag{6}$$

where $\kappa = \frac{w_0}{1-\delta_1}$ is the limit value of the response variable when $t \to \infty$ and $\beta = -\ln(|\delta_1|)$ describes the speed at which the response variable switch from pre-intervention level $\mu_Y^{(0)}$ to stable post-intervention level κ. We have the following connection with β in (3):

Theorem - Suppose that the data-generation process follows a law for $X^{(t)} = (X_1^{(t)}, \ldots, X_m^{(t)})$ given by the parameterization (1) with intervention effect as in (3) for some unknown DAG $G = ([m], E)$ and parameters $\lambda_{kj}, \rho_j^{(t)}, \omega_j^{(t)}, \beta$. Let $X_m^{(t)} \sim \mathcal{N}(\mu_m^{(t)}, \sigma_m^{(t)})$ denote the marginal distribution of $X_m^{(t)}$, and $X^* = (X_1^*, \ldots, X_m^*)$ denote the limiting law of the intervention. Then the parameter values w_0 and δ_1 solving the Box-Tiao model (5) satisfy

$$\mathbb{E}[X_m^*] = \kappa = \frac{w_0}{1 - \delta_1} \qquad \text{and} \qquad \beta = -\ln(|\delta_1|).$$

In words, the Box-Tiao model is recovered as a marginal of the multivariate causal model given by (1) and (3). As noted in Appendix A2, we assume constant effects λ_{ki} along the edges $k \to i$ over time.

c) Parameters Estimations and Causal Interpretations

Parameters - Considering the time series $\tilde{\mu}_m^{(t)}$ as $\mu_m^{(t)}$ centered on pre-intervention data, we have $\tilde{\mu}_m^{(t_i)} = 0$ and we therefore analyze:

$$\tilde{\mu}_m^{(t)} = \begin{cases} n_m^{(t)} & \text{if} \quad t < t_i \\ \tilde{\kappa}(1 - e^{-\beta(t-t_i)}) + n_m^{(t)} & \text{if} \quad t \geq t_i \end{cases} \tag{7}$$

where $\tilde{\kappa} = \kappa - \mu_m^{(t_i)}$. The parameters to estimate are $p = (\tilde{\kappa}, \beta, \phi_1, ..., \phi_p, \Phi_1, ..., \Phi_P, \theta_1, ..., \theta_q, \Theta_1, ..., \Theta_Q)$. The usual procedure is then to fit a SARIMA model on pre-intervention data and then remove it from post-intervention data to retrieve the effect of the intervention term (Aliffia et al., 2024). However, a more straightforward approach consists of using the SARIMA for exogenous variables (SARIMAX) framework, which is based on state-space models (Durbin & Koopman, 2012) and has already been used for intervention analysis (Mphanya et al., 2025). With SARIMAX, it is possible to estimate all parameters except β, which is included inside the intervention term. To overcome this issue, we estimate β numerically by minimizing the likelihood $\mathcal{L}$ of SARIMAX model.

Average Treatment Effects (ATE) - The ATE is computed for each variable j as $\mathbb{E}[X_j^*] - \mathbb{E}[X_j^{(t_i)}]$, which captures the change in the average user behavior due to the intervention. The term $\mathbb{E}[X_j^*] \approx \mu_j^*$ which can be computed using (2) for any variable j. It requires an estimate of ρ_j^*, which can now be obtained using (3), and our β estimate. The term $\mathbb{E}[X_j^{(t_i)}] \approx \rho_j^{(t_i)}$ is one of our MLE outputs. The computation of ATE combines the Box-Tiao estimation with the estimates of the causal parameters to obtain more detailed insights into the long-term effects of the incentive on subpopulations.

Computation steps - To summarize the methodological framework, here are the following steps using data with a set of explanatory variables $X_1, ...X_{m-1}$ and one response variable X_m.

1. *Causal Discovery*: we feed GES algorithm with input data $X_1, ...X_m$ and it estimates the graph $G = (V, [m])$ to which the data distribution is faithful (Maxwell Chickering, 2002).
2. *Estimation of λ and μ*: we find MLE based on Boege et al. (2025) for each time step t.
3. *Estimation of the dynamic model*: we estimate SARIMA parameters $(\phi_1, ..., \phi_p, \Phi_1, ..., \Phi_P, \theta_1, ..., \theta_q, \Theta_1, ..., \Theta_Q)$ and intervention term parameters $\tilde{\kappa}$ and β using the SARIMAX framework (Durbin & Koopman, 2012).
4. *Computation of the ATE*: owing to the β and MLE estimates, we can now compute ATE from which we can draw conclusions on the average user behavior.

3 Results and discussion

a) Data

In 2014, the PTN of Hong Kong implemented a fare incentive policy, which aimed to reduce congestion during peak hours (8:15 to 9:15 am). The incentive consisted of a 25% discount on fare for users tapping out at one of the 29 stations targeted by the policy, between 7:15 and 8:15 am. Consecutive to this policy, Wang et al. (2025) identified 3 behavioral groups:

- *Late Adopters* who used to travel after 8:15 and now adopt the incentive;
- *Early Adopters* who used to travel before 7:15 and now adopt the incentive;
- *Conservatives* whose behaviors don't change.

The data is extracted from Automatic Fare Collection (AFC) systems from July 2014 to October 2015. We select a panel of consistent users for our longitudinal analysis, that is any adult user having more than 10 trips per month at any promotion stations during morning period (6:00 to 10:00 am). Also, we ensure that users have trip records for any three consecutive months over the study period. We end up with a panel of 21,437 users, representing 5,453,047 logs over the study period. Table 1 introduces the user characteristics used for each trip.

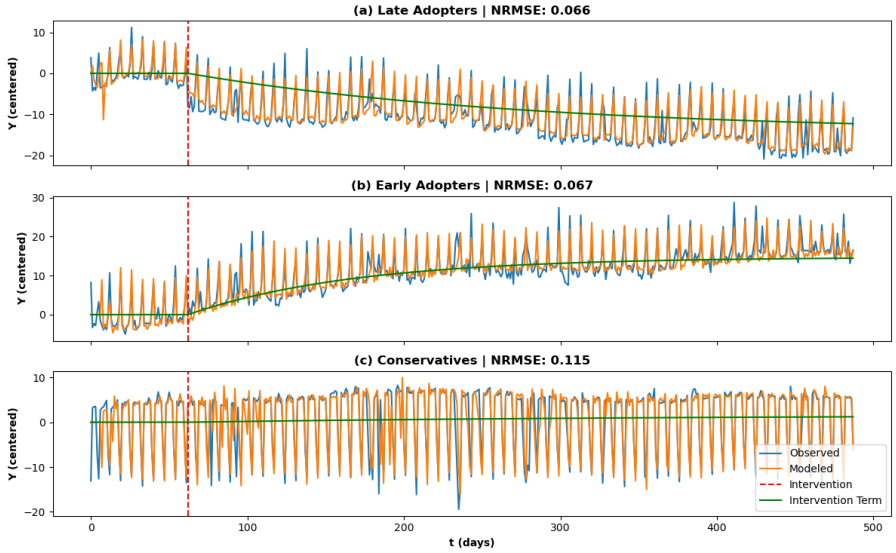
In addition, model performance is evaluated with Normalized RMSE (NRMSE = $\sqrt{\frac{\frac{1}{n}\sum(y-\hat{y})}{y_{\max}-y_{\min}}}$).

Table 1: Description of features

	ID	Description
Tap-out time	*OutTime*	Tap-out time
Percentage of weekdays	*%Weekday*	Number of weekdays with travel / total number of days with travel
Temporal Fidelity	*TempFid*	Average number of days with travel per week
Temporal Flexibility	*TempFlex*	First trip tap-in time standard deviations on weekdays
Spatial Fidelity	*SpaFid*	The percentage of trips that start or end at the station that a user visits most in a month
Spatial Flexibility	*SpaFlex*	The standard deviation of the relative importance (RI is the share of tap-ins made at station $s \in \mathcal{S}$ for user u among all tap-ins of u in $\mathcal{S}$)
Home Fidelity	*HomeFid*	The mean of the number of trips that begin or end at the inferred home station of the user per day in a month
Home Flexibility	*HomeFlex*	The standard deviation of the number of trips that begin or end at the inferred home station of the user per day in a month

b) Model Estimation

Figure 1 shows parameter estimates and displays time-series reconstruction according to the estimated models for the 3 groups Late Adopters, Early Adopters, and Conservatives, respectively modeled with SARIMA(p,d,q)(P,D,Q,s): (1,0,1)(0,1,1,7), (0,0,1)(1,1,1,7) and (0,0,1)(1,1,0,7), plus an intervention term. Results show good performance for late (NRMSE=0.066) and early adopters (NRMSE=0.067), while the performance metric for conservatives are less robust (NRMSE=0.115). All models require seasonal differencing at the week level (D=1, s=7). However, the structure of the auto-regressive and moving average components differs from one model to another.



Late Adopters
$\phi_1 = 0.9687^{**}$
Φ_1 = -
θ_1 = -0.7568**
Θ_1 = -0.9430**
κ = -14.3238**

Early Adopters
ϕ_1 =-
$\Phi_1 = 0.1751^{**}$
$\theta_1 = 0.2781^{**}$
Θ_1 = -1.1417**
κ = 14.7286**

Conservatives
ϕ_1 = -
Φ_1 = -0.4588**
$\theta_1 = 0.3776^{**}$
Θ_1 = -
$\kappa = 1.6252^{ns}$

Figure 1: *Left:* Visualization of true time series $\mu_m^{(t)}$ (blue curve), modeled time series (orange curve), and intervention term (green curve) for each behavioral group. *Right:* Corresponding parameters estimates (right). *Significance level: ** p-value < 0.01, * p-value < 0.05, ns p-value > 0.05*

The core of the modeling approach lies in the evaluation of intervention coefficients. First, κ coefficient is significant for late adopters and early adopters, and non-significant for conservatives. It means that the intervention model is consistent for adopters and rejected for conservatives, which proves the ability of our model to finely capture the effect of the intervention. Moreover, $\kappa > 0$ for early adopters means that the intervention causes an increase in tap-out time, while $\kappa > 0$ for late adopters means that the intervention causes a decrease in tap-out time. It captures well the effect of the intervention on mobility patterns for each group. Second, β is numerically computed and given only if the κ coefficient is significant. Late adopters ($\beta = 0.0046$) display a slower trend than early adopters ($\beta = 0.0094$), suggesting the distribution of adoption time for early adopters is more concentrated toward the intervention time period. This shows a stronger engagement of early adopters compared to late adopters towards taking the incentive, which is not what the fare incentive policy was primarily aimed for.

c) Causal Graph and Average Treatment Effects

Figure 2 shows the causal structure of the data found using the GES algorithm. Causal relationships pointing towards the response variable X_m (tap-out time) involve Percentage of weekdays, Temporal Flexibility, Spatial Flexibility, and Home Flexibility. These results are consistent with Wu et al. (2025), who highlighted the causal role of flexibility in taking the incentive.

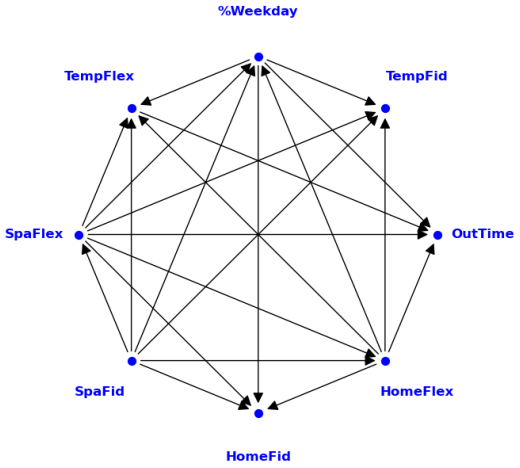


Figure 2: Estimated Causal Graph

Variable j	**Late**	**Early**	$\mu_j^{(t_i)}$
Tap-out time	-14.84	5.05	76.87
TempFid	-3.72	-3.95	5.89
%Weekday	-0.05	-0.01	0.72
TempFlex	-32.19	-52.95	23.07
SpaFlex	0.23	0.25	0.15
SpaFid	0.01	0.00	0.42
HomeFid	-0.02	-0.02	0.95
HomeFlex	-0.44	-0.36	0.12

Table 2: Average Treatment Effects

ATE are shown in Table 2. The sample mean of each variable (including all profiles) is also provided to give an order of its scale. The intervention negatively changed late adopters expected tap-out time (-14.84) and positively changed early adopters expected tap-out time (+5.05). This is consistent with observations as late adopters travel earlier to catch the incentive, and early adopters travel later for the same reason. Also, the intervention caused the average adopter to be more strict in its temporal behavior, reducing its temporal flexibility (late: -32.19; early: -52.95) and also diminishing the consistency of its previous temporal habits captured by their temporal fidelity (late: -3.72; early: -3.95). The incentive led the average user to be more spatially flexible (late: +0.23; early: +0.25) and more regular regarding its home trips (late: -0.44; early: -0.36). To a larger extent, this suggests an effect of the incentive on the purpose of the trip (eg. *would I skip an activity to catch the incentive?*) and the location of activity (*eg. would I leave the the subway further away from work to catch the incentive?*). Answering these questions more finely would require the inference of socio-economic and land-use data.

4 Conclusions

This work investigates the use of causal inference techniques based on graphical modeling for intervention analysis. The novelty of the proposed approach lies in the development of new model for causal behavior response under interventions , based on a re-parametrization of existing models with knowledge from Box-Tiao. Using fare incentives as a case study, the model introduced in this research is able to (1) correctly assess the effect of intervention, (2) evaluate the corresponding intervention term parameters, namely speed of transition (β) and the induced level of response variable (κ), and finally (3) estimate causal structure and causal effects of relevant variable of the response variable. It shows the role of spatio-temporal flexibility in the observed behavioral changes.

This work is a first step towards better controlling and monitoring of public transport systems through user incentivization. Further work could use the versatility of this modeling approach to study the response variable's variance or study other behaviors such as attrition (ie. dropout after incentive adoption). Also, with sufficient data, parameters β and κ can be tuned to orient decision-making in a dynamic and capacity-contained and/or revenue-constrained policy setting. By conditioning on relevant variables, suitable subpopulation groups can be found (eg. most flex-

ible users) to fine-tune the level of incentive which should be proposed, that is, knowing when and to what extent to incentivize or re-incentivize. Our approach is purely data-driven and solely focuses on AFC data, but additional socio-economic and land-use information might significantly improve the accuracy of the model and help to condition on more relevant variables.

Acknowledgements

The authors acknowledge the support of MTR for providing the data for the analysis, and KTH Digital Futures that support the funding of this research.

A Appendices

A1. Solution for the difference equation

From (5) and for $t \geq t_i$, we have the first-order non-homogeneous linear recurrence $\mu_m^{(t)} = \delta_1 \mu_m^{(t-1)} + w_0$, with the non-homogeneous part being constant. The general solution is a combination of the solution to the corresponding homogeneous (H) and a non-homogeneous (P) parts of the recurrence $\mu_m^{(t)} = \mu_m^H + \mu_m^P$, where $\mu_m^P = w_0 + \delta_1 \mu_m^P \Leftrightarrow \mu_m^P = \frac{w_0}{1-\delta_1}$ and $\mu_m^H = C\delta_1^t$, where C is a constant. Using initial conditions, $C = \mu_m^{(t_i)} - \frac{w_0}{1-\delta_1}$, and we have:

$$\mu_m^{(t)} = \begin{cases} 0 & \text{if} \quad t < t_i \\ \mu_m^{(t_i)} \delta_1^t + \frac{w_0}{1-\delta_1}(1-\delta_1^t) + n^{(t)} & \text{if} \quad t \geq t_i \end{cases} \tag{8}$$

Therefore, we can re-parameterize (8) with $\beta = -\ln(|\delta_1|)$ and $\kappa = \frac{w_0}{1-\delta_1}$ to get (6), as stated in the Theorem.

A2. Lambda constant assumption

Under the assumptions stated in the Theorem, we can verify the result by computing the marginal expectation $\mu_m^{(t)}$ of $X_m^{(t)}$ for $t \geq t_i$ in the model given by (1) and (3) and see that the result identifies the parameters δ_1 and w_0 in (6):

$$\begin{aligned}
\mathbb{E}[X_m^{(t)}] &= \rho_m^{(t)} + \sum_{k=1}^{m-1} \sum_{P \in \mathcal{P}(k,m)} \rho_k^{(t)} \prod_{k \to i \in P} \lambda_{ki} \qquad \text{(by (2))} \\
&= \left(\rho_m^{(t_i)} e^{-\beta(t-t_i)} + \rho_m^* (1 - e^{-\beta(t-t_i)}) \right) \\
&\quad + \left(\sum_{k=1}^{m-1} \sum_{P \in \mathcal{P}(k,m)} (\rho_k^{(t_i)} e^{-\beta(t-t_i)} + \rho_k^* (1 - e^{-\beta(t-t_i)}) \prod_{k \to i \in P} \lambda_{ki} \right) \qquad \text{(by (3))} \\
&= \left(\rho_m^{(t_i)} + \sum_{k=1}^{m-1} \sum_{P \in \mathcal{P}(k,m)} \rho_k^{(t_i)} \prod_{k \to i \in P} \lambda_{ki} \right) e^{-\beta(t-t_i)} \\
&\quad + \left(\rho_m^* + \sum_{k=1}^{m-1} \sum_{P \in \mathcal{P}(k,m)} \rho_k^* \prod_{k \to i \in P} \lambda_{ki} \right) (1 - e^{-\beta(t-t_i)}) \qquad \text{(by (2))} \\
&= \mathbb{E}[X_m^{(t_i)}] e^{-\beta(t-t_i)} + \mathbb{E}[X_m^*](1 - e^{-\beta(t-t_i)}) \\
&= \mu_m^{(t_i)} e^{-\beta(t-t_i)} + \mu_m^* (1 - e^{-\beta(t-t_i)}) \qquad \text{(since } \mu_m^{(t)} \text{ is defined as } \mathbb{E}[X_m^{(t)}]).
\end{aligned} \tag{9}$$

When the SARIMA process $n_m^{(t)}$ has mean 0, the Box-Tiao model sets $\mathbb{E}[X_m^{(t)}] = \mu_m^{(t_i)} e^{-\beta(t-t_i)} + \kappa(1 - e^{-\beta(t-t_i)})$, from which it follows that $\kappa = \frac{w_0}{1-\delta_1} = \mu_m^* = \mathbb{E}[X_m^*]$ and $\beta = -\ln(|\delta_1|)$.

References

Aliffia, N., Sediono, S., Suliyanto, S., Mardianto, M. F. F., & Amelia, D. (2024, February). Modeling of world crude oil price based on pulse function intervention analysis approach. *Media Statistika*, *16*(2), 136–147. Retrieved 2026-01-29, from https://ejournal.undip.ac.id/index.php/media_statistika/article/view/51566 doi: 10.14710/medstat.16.2.136-147

Boege, T., Kubjas, K., Misra, P., & Solus, L. (2025, November). Colored Gaussian directed acyclic graphical models. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, qkaf068. Retrieved 2026-01-06, from http://arxiv.org/abs/2404.04024 (arXiv:2404.04024 [math]) doi: 10.1093/jrsssb/qkaf068

Box, G. E. P., & Tiao, G. C. (1975, March). Intervention Analysis with Applications to Economic and Environmental Problems. *Journal of the American Statistical Association*, *70*(349), 70–79. Retrieved 2026-02-08, from https://www.tandfonline.com/doi/full/10.1080/01621459.1975.10480264 doi: 10.1080/01621459.1975.10480264

Durbin, J., & Koopman, S. J. (2012). *Time Series Analysis by State Space Methods: Second Edition*. Oxford University Press. Retrieved 2026-03-26, from https://academic.oup.com/book/16563 doi: 10.1093/acprof:oso/9780199641178.001.0001

Lauritzen, S. L. (1996). *Graphical models* (No. 17). Oxford : New York: Clarendon Press ; Oxford University Press.

Maxwell Chickering, D. (2002). Optimal structure identification with greedy search. *The Journal of Machine Learning Research*, *Volume 3*, Pages 507 – 554. Retrieved from https://jmlr.org/papers/volume3/chickering02b/chickering02b.pdf doi: https://doi.org/10.1162/153244303321897717

Mphanya, A., Mahlangu, S., Masena, T., & Shongwe, S. (2025, July). Seasonal Autoregressive Integrated Moving Average with Exogenous Variables Intervention Analysis: Application to the South African Tourism Industry. *Journal of Statistics Applications & Probability*, *14*(4), 507–523. Retrieved 2026-01-29, from https://www.naturalspublishing.com/Article.asp?ArtcID=30956 doi: 10.18576/jsap/140402

Rios, F. L., Moffa, G., & Kuipers, J. (2025). **Benchpress** : A Versatile Platform for Structure Learning in Causal and Probabilistic Graphical Models. *Journal of Statistical Software*, *114*(12). Retrieved 2026-03-26, from https://www.jstatsoft.org/v114/i12/ doi: 10.18637/jss.v114.i12

Wang, L., Chen, X., Ma, Z., Zhang, P., Mo, B., & Duan, P. (2025, February). Data-driven analysis and modeling of individual longitudinal behavior response to fare incentives in public transport. *Transportation*, *52*(1), 263–286. Retrieved 2026-01-14, from https://link.springer.com/10.1007/s11116-023-10419-8 doi: 10.1007/s11116-023-10419-8

Wu, Y., Markham, A., Wang, L., Solus, L., & Ma, Z. (2025). Data-driven causal behaviour modelling from trajectory data: A case for fare incentives in public transport. *Journal of Public Transportation*, *27*, 100114. Retrieved 2025-11-28, from https://linkinghub.elsevier.com/retrieve/pii/S1077291X24000341 doi: 10.1016/j.jpubtr.2024.100114

Zhou, F., Li, C., Huang, Z., Xu, R., & Fan, W. (2022, March). Fare incentive strategies for managing peak-hour congestion in urban rail transit networks. *Transportmetrica A: Transport Science*, *18*(1), 166–187. Retrieved 2026-03-27, from https://www.tandfonline.com/doi/full/10.1080/23249935.2020.1793028 doi: 10.1080/23249935.2020.1793028