Sensitivity Analysis of Traffic Equilibria with Applications to OD-Estimation

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Sensitivity analysis of traffic equilibria with respect to e.g. link cost parameters or OD-demands is an important area. Early constructive work on this problem, based on a reduced set of equilibrium paths, and employing matrix techniques was Tobin and Friesz (1988). Their results were corrected in Yang and Huang (2005). Another line of approach, where an auxiliary equilibrium problem is solved to get sensitivities, is represented by Patriksson and Rockafellar (2003), and Patriksson and Rockafellar (2004).

A recent application of the Yang and Huang approach to OD estimation is Lundgren and Peterson (2008). They compute flow sensitivities with respect to OD demands by solving a quadratic programming problem for each OD pair.

We, Lindberg and Engelson (2013), have recently given closed-form expressions for sensitivities w.r.t. tolls in multi-class traffic equilibria. This analysis carries over to the single-class non-tolled case in a straightforward way.

The derivations are based on equilibrium cycles, i.e. flow bearing cycles with reduced cost zero in the shortest path determination for some OD-pair at equilibrium. Let the cycles be represented by their arc-indicator vectors, with a +1 for an arc along the cycle direction, a -1 for an arc against, and 0 for arcs not in the cycle.

Further let $\mathcal{C} = \{\mathbf{g}_i\}_{i\in I}$ be a basis of the equilibrium cycles. In a generic case flow can be sent in both directions along these cycles, otherwise we are in a situation with appearing or disappearing cycles, where we do not have differentiability. Let $\mathbf{G}$ be the matrix with columns $\mathbf{g}_i$.

Suppose that the link travel times are of the form $t_a(f_a, \mathbf{p})$, where f_a is the link flow in link $a \in \mathcal{A}$, and $\mathbf{p}$ is a parameter vector. Let $\mathbf{t}(\mathbf{f}, \mathbf{p})$ and $\mathbf{f}$ be the corresponding vectors.

When $\mathbf{p}$ is changed, the OD-flows adjust to keep the times of utilized routes equal. This corresponds to sending flow along the equilibrium cycles to keep their costs at 0.

So suppose $\mathbf{p}$ is changed from $\overline{\mathbf{p}}$ to $\overline{\mathbf{p}} + d\mathbf{p}$. Then the OD flows will adjust to keep the costs of the cycles at 0. They will then adjust by decreasing flow on the route with increased times, and vice versa. This corresponds to sending flow along the cycles.

Let $d\boldsymbol{\gamma}$ be the vector of total flow changes along the cycles. Then the change in link flows is $d\mathbf{f} = \mathbf{G}d\boldsymbol{\gamma}$. The corresponding link time change becomes $dt_a = \frac{\partial t_a}{\partial f_a} df_a + \nabla_p t_a d\mathbf{p}$, giving the total link time change $d\mathbf{t} = \nabla_f \mathbf{t} d\mathbf{f} + \nabla_p \mathbf{t} d\mathbf{p} = \nabla_f \mathbf{t}\mathbf{G} d\boldsymbol{\gamma} + \nabla_p \mathbf{t} d\mathbf{p}.$

Before the change we have equilibrium, i.e. costs around the cycles are zero. That is, for cycle i,

$$\mathbf{g}_i^T \mathbf{t} = 0. \qquad (1)$$

After the change we still have equilibrium, which in view of (1) gives

$$0 = \mathbf{g}_i^T d\mathbf{t} = \mathbf{g}_i^T (\nabla_f \mathbf{t}\mathbf{G} d\boldsymbol{\gamma} + \nabla_p \mathbf{t} d\mathbf{p}). \qquad (2)$$

Collecting the equations in (2) into one system we get

$\mathbf{0} = \mathbf{G}^T d\mathbf{t} = \mathbf{G}^T \left(\nabla_f \mathbf{t}\mathbf{G} d\boldsymbol{\gamma} + \nabla_p \mathbf{t} d\mathbf{p}\right)$, giving

$$d\boldsymbol{\gamma} = -(\mathbf{G}^T \nabla_f \mathbf{t}\mathbf{G})^{-1} \mathbf{G}^T \nabla_p \mathbf{t} d\mathbf{p}. \qquad (3)$$

This in turn gives

$d\mathbf{f} = \mathbf{G}d\boldsymbol{\gamma} = -\mathbf{G}(\mathbf{G}^{\boldsymbol{T}}\nabla_f\mathbf{tG})^{-\mathbf{1}}\mathbf{G}^{\boldsymbol{T}}\nabla_p\mathbf{t}d\mathbf{p}$, and thus

$$\nabla_p\mathbf{f} = -\mathbf{G}(\mathbf{G}^{\boldsymbol{T}}\nabla_f\mathbf{tG})^{-\mathbf{1}}\mathbf{G}^{\boldsymbol{T}}\nabla_p\mathbf{t}. \quad (4)$$

This is the desired closed form formula for the sensitivities of the link flows w.r.t. the parameter vector **p**.

Elastic demand case

Yang and Huang (2005) derive sensitivities for the case with elastic demand. This is however easily subsumed under the above analysis, by introducing dummy links, as described e.g. in Patriksson (2004), section 2.2.4.

OD estimation

At the surface the OD estimation case does not seem to fit in the above scheme of link time changes, but with some observations, it does.

When the OD demands change, the added (say) demand has to find its way through the network. In a generic situation, it then has to use existing shortest paths. And then it does not matter which path it takes, the existing flows will accommodate to keep the costs of the cycles at zero. So let's single out a unique shortest path $\mathbf{r}_w$ for each OD pair $w \in \mathcal{W}$. We identify the path with its link incidence vector. Sending the (positive or negative) demand change dd_w along the path entails changing the flows in the links of $\mathbf{r}_w$ by dd_w. This added flow gives a marginal change of link times equal to $t'_a(f_a)dd_w$, in that link. Thus, we may consider d_w as a parameter *p* having these marginal effects in the links of $\mathbf{r}_w$.

Hence, the contribution to the link travel time vector **t** is $d\mathbf{t}_w =_{df} \nabla\mathbf{tr}_w dd_w$, and with d**d** being the marginal change in OD demands and **R** the matrix with columns $\mathbf{r}_w$, $d\mathbf{t} = \nabla\mathbf{tR}d\mathbf{d}$. Thus, with **d** as the parameter vector, we get $\nabla_p\mathbf{t} = \nabla\mathbf{tR}$. Combining with (4) we get

$$\nabla_p\mathbf{f} = -\mathbf{G}(\mathbf{G}^{\boldsymbol{T}}\nabla_f\mathbf{tG})^{-\mathbf{1}}\mathbf{G}^{\boldsymbol{T}}\nabla_p\mathbf{t} = -\mathbf{G}(\mathbf{G}^{\boldsymbol{T}}\nabla_f\mathbf{tG})^{-\mathbf{1}}\mathbf{G}^{\boldsymbol{T}}\nabla\mathbf{tR}\,, \quad (5)$$

which is the desired formula.

In OD estimation one usually tries to minimize some combined deviation measure, weighting the deviations of computed link flows from those measured in a subset of the links, against that of **d** from some forecasted values, see e.g. Lundgren and Peterson (2008). Then formula (5) gives the possibility to compute gradients of the combined deviation measure, which in turn can be used in descent schemes for the combined deviation measure

In the talk we will hopefully report on computational experience on a small example.

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