TO MSA OR NOT MSA?
SYSTEM CONVERGENCE IN THE DANISH NATIONAL MODEL

Jeppe Rich, Otto Anker Nielsen
Department of Transport, Technical University of Denmark
Bygningstorvet 116 Vest, 2800 Kgs. Lyngby, Denmark
Emails: jr@transport.dtu.dk, oan@transport.dtu.dk

ABSTRACT

In the paper we investigate the system convergence between the demand and supply system in a large scale transport model for Denmark. The demand is represented by the people that travel in the transport system and on the defined network, whereas the supply is represented by the level-of-service attributes (e.g., travel time and cost) offered to travellers.

For large scale models an “internal approach”, where the traffic assignment is integrated with the demand model, is not in practical possible. As a result we need to carefully consider the convergence of the “external loop” between the two models towards a common equilibrium. As iterations in the external loop are very costly from a computational point of view, there is a good payoff if convergence speed can be improved. Or put in even stronger words, if a sensible convergence cannot be obtained within relative few iterations, the model will be close to useless.

The paper is a continuation of work presented in Rich et al. (2010) based on a toy-network. In this paper we investigate large scale performance in the newly implemented Danish National Model with 907 zones, more than 15 travel segments and state-of-the-art stochastic user equilibrium assignment models. In the paper we compare the method of repeated approximations (MRA) with different variants of the Method of Successive Averages (MSA).

2. INTRODUCTION TO FIX POINT ALGORITHMS

If we let demand be represented as a continuous vector function $\boldsymbol{D}(\boldsymbol{x}) : S \subseteq R^N \rightarrow S$ and S a non-empty, compact and convex set. By Brouwer theorem it has at least one fixed-points $\boldsymbol{x}^* = \boldsymbol{D}(\boldsymbol{x}^*)$ (existence). If it can also be proved that at most a fixed-point exist (uniqueness), the unique fixed-point may be found through many algorithms, whose general specification can be written as;

(1) $$\boldsymbol{x_k} = \boldsymbol{x_{k-1}} + \boldsymbol{M_k}\,(\boldsymbol{D}(\boldsymbol{x_{k-1}}) - \boldsymbol{x_{k-1}})$$

The algorithm is based on a starting solution $\boldsymbol{x_0} \in \boldsymbol{S}$, and a matrix $\boldsymbol{M_k}$ (see among many others Kelley, 1995).

The method of repeated approximations (MRA), as in the Banach theorem, is given by $\boldsymbol{M_k} = \boldsymbol{I}.$ It is generally recognised that MRA, which is basically a simple iteration scheme where the level-of-service variables is fed directly to the route-choice and vice versa, may exhibit an unstable pattern and lead to cyclic unstable solutions. It can be shown that the contraction region, e.g. the region for which a starting solution $\{x, t\}$ will render stable convergence, depends on the slope of the demand and supply curve. Generally, as the slope between the curves increases, the contraction region shrinks.

The method of successive averages (MSA) is given by $\boldsymbol{M_k} = a_k\boldsymbol{I}$ with $a_k = 1/k$, and convergence conditions are given by the Blum theorem (Blum, 1954) or by an extension in Cantarella (1997). The "success" of the MSA (and variants of the MSA) in the sense that it converge is due to the fact that "averaging" form a contraction principle.

Interesting variants of the MSA include the "weighted MSA" approach was considered in Liu et al. (2009) where a_k is given by

(2) $$a_k = \frac{k^d}{1^d + 2^d + 3^d + \cdots + k^d}$$

Where $d \geq 0$. Clearly, for $d = 0$ the ordinary MSA emerge, however, as d increases more emphasis is put on the latest iterations.

Below we present a sequence of results from a number of model runs where we compare a simple MRA with a weighted MSA where $d = 2$. In the runs we measure the difference between two subsequent iterations by measuring the distribution of the difference. To stress-test the model in the following we have applied a 2040 population and reduced car costs by 25%. More specifically, for Figure 1, which compares iteration 1 with iteration 2, we calculate for both methods the relative deviation at the level of the cells. These deviations are then plotted as a density function so that we can easily see how the tails are behaving. It should be said that the final histogram has been created as a weighted histogram, where the different cells are weighted according to number of trips.

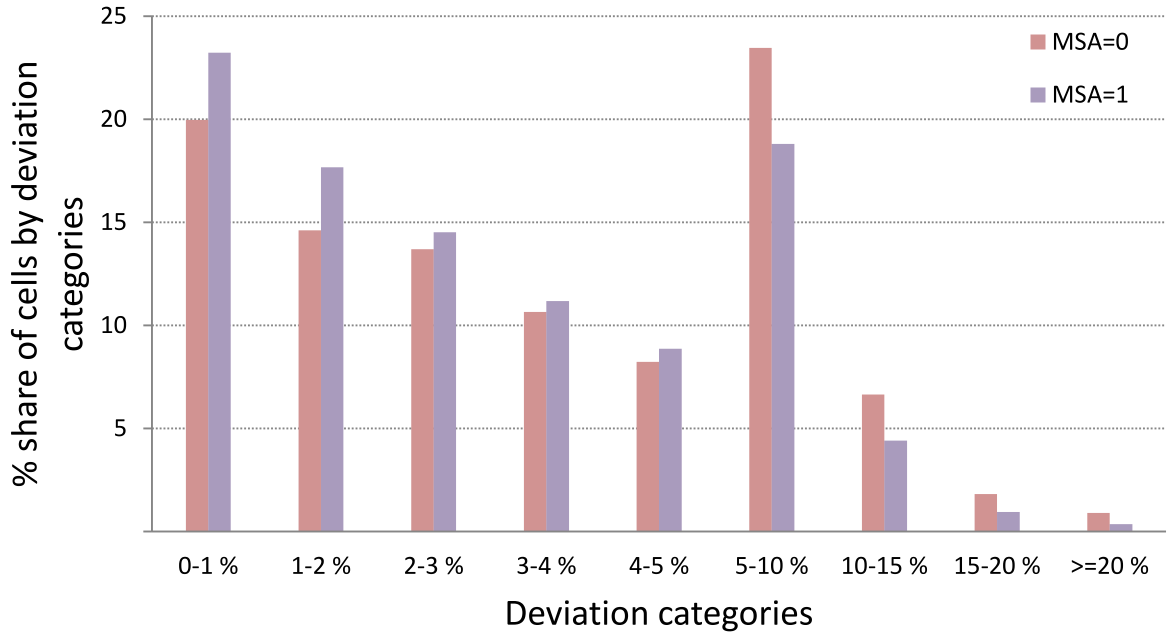

Figure 7: Distribution of deviation of OD matrix cells between iteration 1 and 2 in the outer loop. The distribution is weighted with average number of trips in cells.

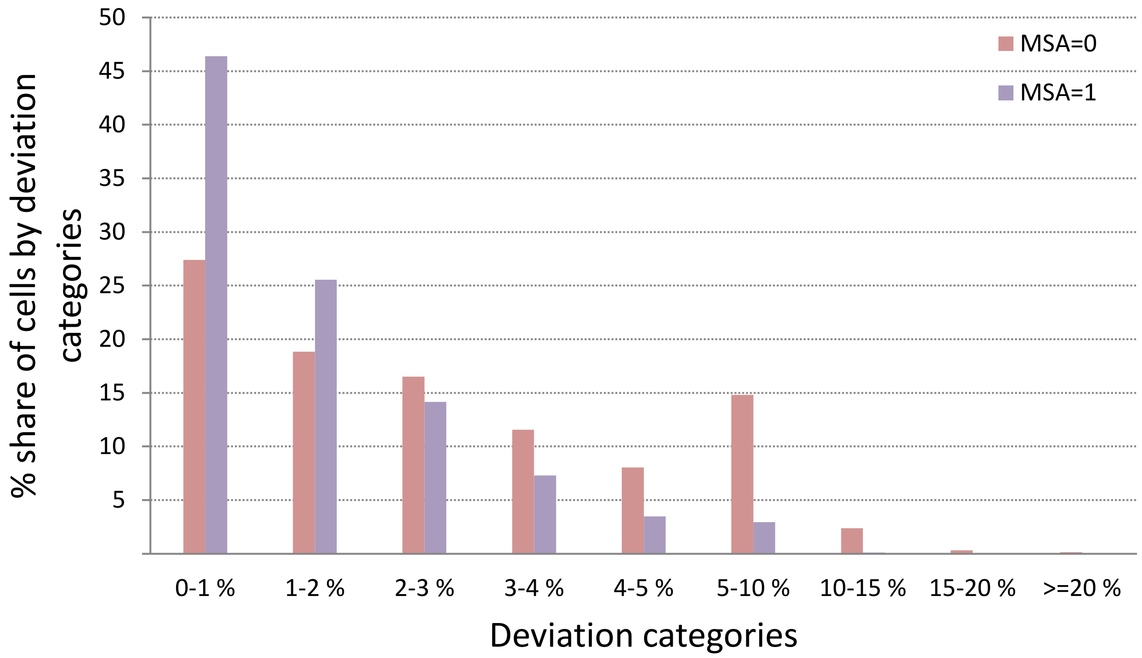

Figure 7: Distribution of deviation of OD matrix cells between iteration 2 and 3 in the outer loop. The distribution is weighted with average number of trips in cells.

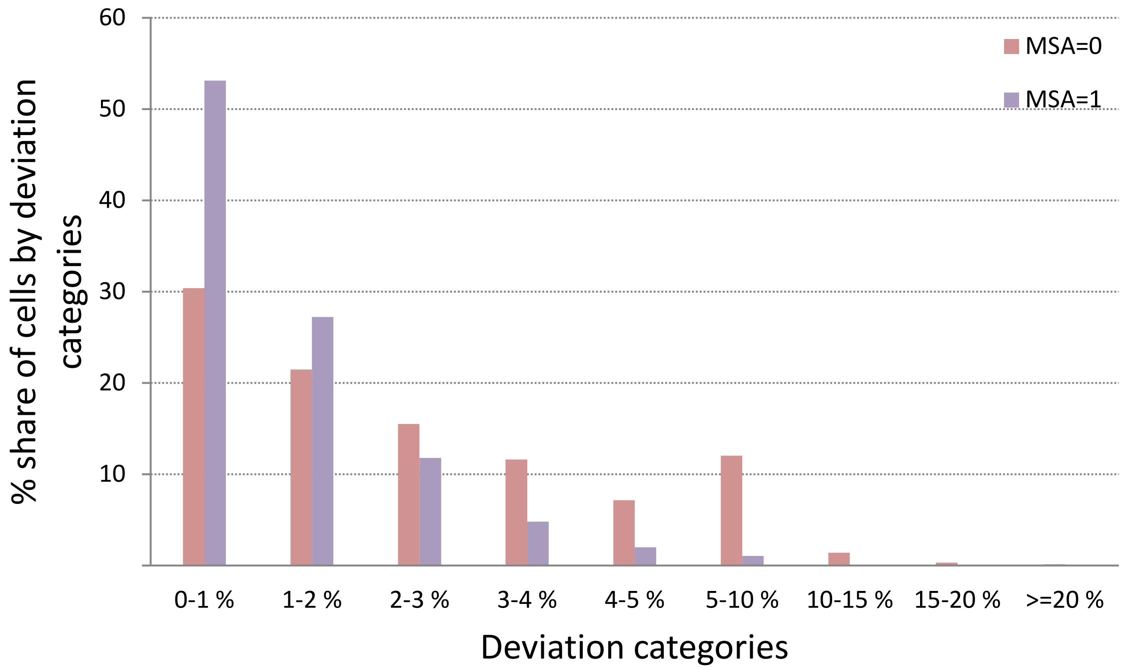

Figure 7: Distribution of deviation of OD matrix cells between iteration 3 and 4 in the outer loop. The distribution is weighted with average number of trips in cells.

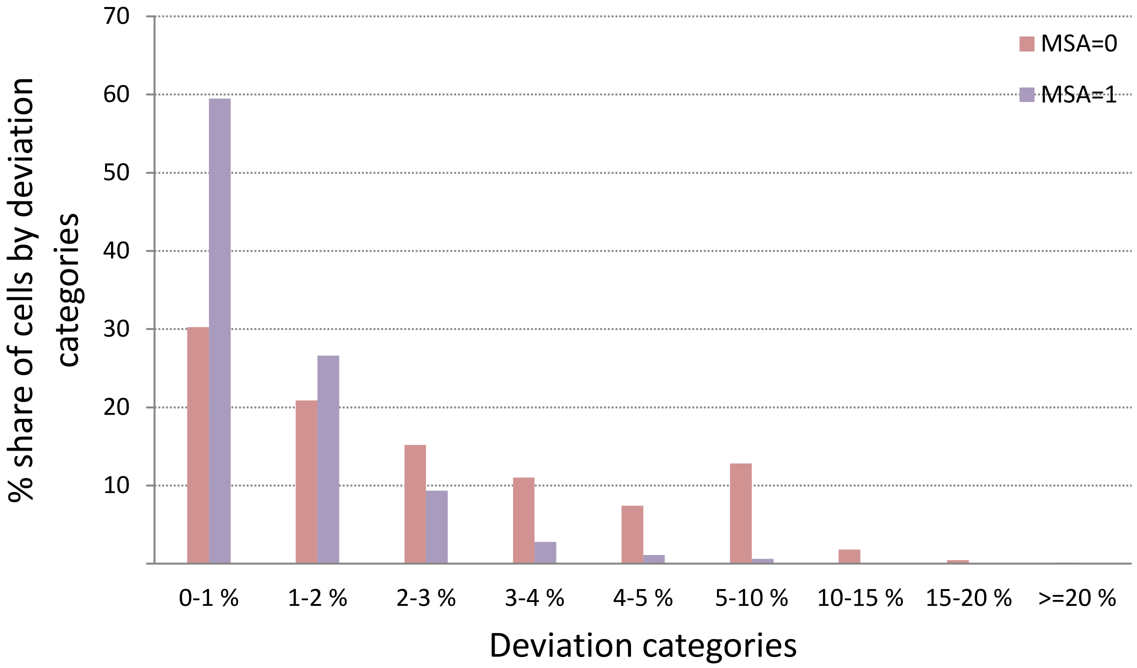

Figure 7: Distribution of deviation of OD matrix cells between iteration 4 and 5 in the outer loop. The distribution is weighted with average number of trips in cells.

As clearly seen, the results are very strongly in favour of the weighted MSA. The difference between the two methods is not very significant between iteration 1 and 2. However, as the iteration scheme progress the difference becomes larger and larger. In fact, it is hard to argue that the MRA show any sign of finite convergence and the number of cells with deviation between 5-10% stays around 12-14%, which is not acceptable for most applications. The deviation of the MSA on the other hand, works quite well. It should be noted that as the car assignment is stochastic some uncontrolled noise will always be inherited in the final matrices.

6. LITERATURE

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