# Uniqueness of Stochastic User Equilibrium on urban road networks with realistic volume-delay functions


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## Aim of the paper

The main aim of this paper is to describe sufficient conditions for the uniqueness of Stochastic User Equilibrium and the convergence of solution algorithms weaker than the traditional ones, including non-monotone non-separable arc cost functions whose Jacobian may be asymmetric and/or non positive semi-definite; a preliminary discussion of these issues is in [4]. The need to address this apparently unusual set-up for an assignment model stems from the fact that the representation of congestion in urban networks allowed by the standard uniqueness conditions, such as the monotonicity of separable volume-delay functions, is not realistic and thus may lead to wrong decision in the planning process. Indeed, the main delays suffered by drivers when links are short derive from the intersections, where vehicle flows conflict, competing to use the capacity of links ahead (merging), or are slowed by other vehicles that are queuing (diversion). These traffic phenomena cannot be modelled through separable cost functions, nor with symmetric Jacobians. A suitable supply model is then introduced and weak sufficient conditions are applied, showing that to ensure uniqueness of the equilibrium there is a trade-off between congestion level and perception error in the route choice.

## Introduction

Static assignment is still the most widely used paradigm for simulating transport networks and their interaction with travel demand flows (e.g. [3]).

The supply side of the equilibrium model should represent travel times as a function of vehicle flows. However, we sadly have to admit (e.g. [7]) that the volume-delay functions currently applied in practice and implemented in the available software are not yet capable to reproduce the most relevant congestion phenomena occurring in urban networks: vehicle stream conflicts and queues.

The strong appeal of (scalar) polynomial vdf-s, such as the well known BPR, is due in part to their simplicity, but also to the lack of valid alternatives. Traffic assignment models have been often conceived in the field of Operation Research instead that in the field of Transport Engineering, so the aim of applying advanced optimization algorithms on large network problems prevailed on the need of realistic results. For many years, the common mathematical framework where to cast user equilibrium problems has been nonlinear optimization and variational inequalities (e.g. [8]), while only more recently the highly flexible theory of fixed-points started to be widely applied in traffic assignment formulations (e.g. [5], [2], [1]).

For optimization based formulations, the available uniqueness condition requires that the Jacobian of the volume-delay function is positive definite when route choice of users is deterministic. Unfortunately, this only apparently extends the range of applicable supply models from separable (the travel time on a link depends only on the flow of that link) to asymmetric arc cost (vector) functions (the travel time depends also on the flows of other adjacent links, and not reciprocally). Indeed, in our knowledge only few relevant phenomenon in urban traffic can be suitably reproduced through a non-separable volume-delay function whose Jacobian is positive definite.

For fixed-point based formulations, we showed in some earlier work (e.g. [4]) that, when the route choice of users is stochastic, there exists a wider assortment of uniqueness results which extend the classical sufficient conditions, thus paving the way to new modelling opportunities on the supply side of the equilibrium problem.

In particular, we will exploit the sufficient condition proposed in [6] stating that the stochasticity of the route choice allows to determine a positive scalar $\alpha$, such that, if the Jacobian of the arc cost function summed to the identity matrix multiplying $\alpha$ is positive definite, then the SUE problem has unique solution.

To this end we will introduce a non-separable vdf specifically conceived to reproduce vehicle stream conflicts and queues, although in a static framework. Then we will apply the above condition to the proposed supply model, showing that to ensure uniqueness of the equilibrium there is a trade-off between congestion level and perception error in the route choice.

## Formulation and results

In this section we present a preview of some results contained in the paper, where their detailed explanation is provided.

By combining the path choice map P(·) with the arc cost function c(·), SUE is formulated through the following fixed-point problem in the space of feasible arc flows $S_f$ :

$$\boldsymbol{f} = \boldsymbol{\Delta}\cdot\boldsymbol{Q}\cdot\mathrm{P}(\boldsymbol{\Delta}^{\mathrm{T}}\cdot\mathrm{c}(\boldsymbol{f})),\ \ \boldsymbol{f}\in S_f\,. \qquad (1)$$

where $\boldsymbol{\Delta}$ is the arc-path incidence matrix, whose generic element $\delta_{ak}$ is 1, if path $k\in K$ includes arc $a\in A$, and is 0, otherwise, while $\boldsymbol{Q}$ is the diagonal demand matrix whose $k$-th element is equal to the flow $Q_{od}$ between the origin destination couple ($o$, $d$) that path $k\in K_{od}$ connects.

Assume that the path choice of the generic ($o$, $d$) couple is given by a Logit model with parameter $\theta_{od}$ . In the paper we prove that the following is a sufficient condition for the uniqueness of SUE:

$$\nabla\mathrm{c}(\boldsymbol{f}) + \alpha_{\mathrm{LB}}\cdot\boldsymbol{I}_{|A|} > 0,\ \forall\boldsymbol{f}\in S_f\,,\quad \alpha_{\mathrm{LB}} = \theta_{od} / (4\cdot m\cdot Q_{od})\,, \qquad (2)$$

where $m$ is the number of links of the longest path in $K$.

Moreover, we propose the following non-separable volume-delay function. Let $T$ be the duration of the peak hour, $L$ be the average length of the trip, $L_a$ , $q_a$ , $p_a$ and $g_a$ be the length, the capacity, the priority and the green share of the generic arc $a\in A$, respectively, and $\mu$ be a calibration parameter. We assume that the intersection delay, for diversions and signalized mergings is:

$$t_a = 0.5\cdot\mu\cdot T / L\cdot L_a / q_a\cdot(\,f_a - \min\{\,f_a\,,\,q_a\cdot g_a\,,\,q_b\cdot g_a\cdot f_a / f_{ab}\,;\,\forall b\in FS(a)\})\,, \qquad (3a)$$

while for unsignalized mergings it is:

$$t_a = 0.5\cdot\mu\cdot T / L\cdot L_a / q_a\cdot(\,f_a - e_a)\,, \qquad (3b)$$

where $e_a$ can be obtained through the following iterative algorithm:

$e_a \leftarrow 0\ ,\forall a\in BS(b)$

**loop**

  $r \leftarrow q_b - \sum_{a\in BS(b)} e_a\ ;\ s_a \leftarrow f_a - e_a\ ,\forall a\in BS(b)\ ;\ B \leftarrow \{a\in BS(b)\colon s_a > 0\}$

  **if** $B \neq \emptyset$ **then exit loop**

  $e_a \leftarrow e_a + \min\{s_a\,,\,r\cdot(q_a\cdot p_a) / (\sum_{c\in B} q_c\cdot p_c)\}\ ,\ \forall a\in B$

Applying condition (2) to the cost function (3), we have results like:

$\theta_{od} / T > 2 \cdot \mu \cdot Q_{od} / q_a$ ,

showing that the highest is perception error in the route choice the highest is the congestion level for which we can still ensure the uniqueness of the equilibrium.

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